

Galois theories for q -difference equations: comparison theorems*

Lucia Di Vizio[†] and Charlotte Hardouin[‡]

March 3, 2013

Abstract

We establish some comparison results among the different Galois theories, parameterized or not, for q -difference equations, completing [CHS08] and [DVH11b]. Our main result is the link between the abstract parameterized Galois theories, that give information on the differential properties of abstract solutions of q -difference equations, and the properties of meromorphic solutions of such equations. Notice that a linear q -difference equation with meromorphic coefficients always admits a basis of meromorphic solutions.

In the last part of the paper we consider the behavior of the various Galois groups when q is a parameter to be specialized.

Contents

1	The tannakian category of q-difference modules	4
1.1	Fiber functors associated with a Picard-Vessiot extensions	4
1.2	Fiber functor associated with a meromorphic basis of solutions	5
1.3	The forgetful functor and the generic Galois group	6
2	The differential tannakian category of q-difference modules	7
2.1	Differential tannakian structure of $Diff(\mathcal{F}, \sigma_q)$	8
2.2	Formal differential fiber functor	9
2.3	Differential fiber functor associated with a basis of meromorphic solutions	10
2.4	Parameterized generic Galois group	11
3	Comparison of Galois groups	12
3.1	Differential Picard-Vessiot groups over the elliptic functions	14
3.2	Comparison results for generic Galois groups	15
4	Specialization of the parameter q	19
4.1	Specialization of the parameter q and localization of the generic Galois group	19
4.2	Upper bounds for the generic Galois group of a differential equation	23

Introduction

The Galois theory of difference equations have witnesses a major development in the last decade. Nowadays there are many Galoisian approaches to q -difference equations. First of all, the more “classical” Picard-Vessiot theory in [vdPS97]. Then, based on the existence of meromorphic solutions, we have Sauloy Galois group constructed in [Sau04b] and the weak Picard-Vessiot in [CHS08]. The comparison between those theory is studied in [CHS08] or, in certain cases, can be deduced from basic tannakian

*Work partially supported by the grants ANR-06-JCJC-0028 qDIFF and ANR 2010 BLAN-0115-01 HAMOT .

[†]Lucia DI VIZIO, Laboratoire de Mathématiques UMR 8100, CNRS, Université de Versailles-St Quentin, 45 avenue des États-Unis 78035 Versailles cedex, France. divizio@math.cnrs.fr

[‡]Charlotte HARDOUIN, Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 9, France. hardouin@math.univ-toulouse.fr

considerations. In [DV02] and [DVH11a], the authors consider also a generic (called intrinsic by some authors) Galois group in the spirit of [Kat82]. A Galois theory for parameterized families of q -difference equations have been constructed in [HS08]. For such families we also have basis of meromorphic solutions that leads to a weak Picard-Vessiot ring and therefore to another parameterized Galois theory. Moreover generic parameterized Galois groups have been studied in [DVH11b] and a nonlinear Galois theory for q -difference equations has been introduced in [Gra]. The comparison between the two last theories is studied in [DVH11b]. In the present paper we establish some comparison results, complementary to [CHS08] and [DVH11b], completing the picture. Our main result is the possibility of relating the abstract Galois groups mentioned above, that *a priori* give only information of properties of abstract solutions, to actual properties of meromorphic solutions. In the last part of the paper, we consider the case when q is a parameter and we study the behavior of those Galois groups with respect to the specialization of the parameter q , extending some previous results by Y. André [And01].

A function f is hypertranscendental over a field F equipped with a derivation ∂ if $F[\partial^n(f), n \geq 0]$ is a transcendental extension of infinite degree, or equivalently, if f is not a solution of a nonlinear algebraic differential equation with coefficients in F . The question of hypertranscendence of solutions of functional equations appears in various mathematical domains: in special function theory (see for instance [LY08], [Mar07] for the differential independence ζ and Γ functions), in enumerative combinatorics (see for instance [BMP03] for problems of hypertranscendence and D -finiteness¹ specifically related to q -difference equations). In [BMP03], the authors consider some formal power series generated by enumeration of random walks with constraints. Such generating series are solutions of q -difference equations: one natural step towards their rationality is to establish whether they satisfy an algebraic (maybe nonlinear) differential equation. In fact, as proven by J.-P. Ramis, a formal power series which is solution of a linear differential equation and a linear q -difference equation, both with coefficients in $\mathbb{C}(x)$, is necessarily rational (see [Ram92]). Other examples of q -difference equations for which it would be interesting to establish hypertranscendence are given in [BMF95] and [BM96]. A galoisian approach of such questions has been developed in [HS08]. Using Picard-Vessiot machinery, the authors attach to a linear difference equation a differential Galois group *a la Kolchin* i.e. a sub-group of the group of invertible matrices of given order defined by a set of non-linear differential equations. By Galois correspondence, the differential dimension of their Galois group measures the hypertranscendence properties of a basis of formal solutions of the initial difference equation. Using the results of Cassidy on the classification of differential algebraic groups (see [Cas72]), Hardouin and Singer were thus able to prove on a purely algebraic way some hypertranscendence results on classical functions, solutions of functional equations.

However, a first difficulty appears when, for a given difference equation, it comes to comparing the formal solutions to the classical functions which satisfy the same difference equation. Even though some examples are treated in [HS08], no general answer is proposed. This question is nevertheless crucial in the Galois theory of difference equation. For instance, the digamma function $\psi(x)$, i.e. the logarithmic derivative of the Gamma function satisfies

$$(0.1) \quad \psi(1-x) = \psi(x) + \frac{\pi}{\tan(\pi x)}$$

but it is still an open question to establish the connection between the Galois group of the equation (0.1) and the algebraic or differential behavior of the digamma function. This difficulty mainly arises from the fact that (0.1) underdetermines $\psi(x)$. However, for q -difference equations, a result of Praagman ([Pra86]) states that a linear q -difference equation with meromorphic coefficients admits a basis of meromorphic solutions over the field of elliptic functions C_E with respect to the elliptic curve $E = \mathbb{C}^*/q^{\mathbb{Z}}$. In order to compare the differential behavior of these meromorphic solutions and the differential dimension of the differential Galois group of Hardouin-Singer, we choose to use the differential tannakian formalism developed by A. Ovchinnikov ([Ovc09a]). We establish first a correspondence between the distinct notion of solutions related to Picard-Vessiot constructions and the functorial point of view of the differential tannakian category. This categorical framework allows us to compare the analytic and the algebraic constructions and to show that, for q -difference equations, one can descend the differential Galois group of Hardouin-Singer from the differential closure of the field of elliptic function to C_E itself, without

¹A function f is D -finite over a differential field (\mathcal{F}, ∂) if it is solution of a linear differential equation with coefficients in \mathcal{F} .

loosing any information *i.e.* we prove that the differential behavior of the meromorphic solutions is entirely determined by the differential Galois group of Hardouin-Singer (see Theorem 3.5). This result is the differential analogue of [CHS08, Theorem 3.1]. Since a similar result of Praagman holds for the shift difference operator $x \mapsto x + \tau$, the τ -analogue of our proofs will show that the differential behavior of the meromorphic solutions of a τ -difference operator over the τ -periodic functions is controlled by the differential Galois group of Hardouin-Singer. Thus we can give a positive answer to the incarnation of the formal solutions in the case of a q -difference or a τ -difference operator.

Another difficulty of the theory of Hardouin-Singer is the field of definition of their Galois groups. In general, the authors of [HS08] need to work over a differentially closed field *i.e.* an enormous field, to ensure that their Galois groups will have enough points. In [DVH11a], we attached to a linear q -difference equation a differential algebraic group called parametrized generic Galois group whose field of definition is the one of the coefficients of the q -difference equation. Moreover, we gave an arithmetic characterization of this group showing that it is differentially generated by a countable set of operators namely the curvatures of the equation. This last result solved entirely the Grothendieck conjecture for q -difference equations. Using our the curvature characterization, we now prove that the parametrized generic Galois group becomes isomorphic to the differential Galois group of Hardouin-Singer over a suitable field. This result allows to descend the theory of Hardouin-Singer from the differential closure of the field of constants to the field of definition of the coefficients of the equation, whose constants may not even be algebraically closed² (see Corollary 3.12). But, it also resumes the computations of the differential Galois group to an intrinsic computation. This comparison also leads to the first path between the theory of B. Malgrange and the theory of Kolchin, answering a question of B. Malgrange [Mal09, page 2]. In fact, A. Granier, in the wake of B. Malgrange, proposed a Galois theory for non-linear q -difference equations using as Galois group a D -groupoid in the space of jets (in [Gra]). In [DVH11b], we show, using one more time our curvature characterization, that the Malgrange-Granier D -groupoid of a linear q -difference equation gives the parametrized generic Galois group of the equation and therefore, combined with our last comparison result, the differential Galois group of Hardouin-Singer. The fact that one find a differential algebraic group is quite natural since the construction of the Malgrange D -groupoid involves the linearizations of the q -difference equation which are obtained by derivating the equation. However, the main difficulty in the comparison between Malgrange's theories and Kolchin's theories these comparisons was to make the connection between a purely analytic object, the Malgrange D -groupoid defined over an analytic variety and a purely algebraic construction, the Hardouin-Singer Galois group, defined over a differentially closed field. The connection between these two last objects is the parametrized generic Galois group.

At last we compare the generic, algebraic and parametrized Galois group to the generic Galois groups obtained by specialization of q . Inspired by the work of André ([And01]), we prove that the specialization of the algebraic (resp. parametrized) generic Galois group of a q -difference equation $Y(qx) = A(q, x)Y(x)$ with coefficients in a field $k(q, x)$ such that $[k : \mathbb{Q}] < \infty$ at $q = a$ for any a in the algebraic closure of k , contains the generic algebraic (resp. parametrized) Galois group of the specialized equation. If k is a number field, this holds also if we reduce the equations in positive characteristic, so that q reduces to a parameter in positive characteristic. So if we have a q -difference equation $Y(qx) = A(q, x)Y(x)$, we can either reduce it in positive characteristic and then specialize q , or specialize q and then reduce in positive characteristic. In particular, for $q = 1$ we obtain from

$$\frac{Y(qx) - Y(x)}{(q-1)x} = \frac{A(q, x) - 1}{(q-1)x} Y(x)$$

a differential system. In this way we obtain many q -difference systems that either reduce to a differential system or to its reduction modulo p . All these parameterized families of systems are compatible and the associated generic Galois groups contain the generic Galois group of the differential equation at $q = 1$. On the other hand starting from a linear differential equation, one could express the generic Galois group of the differential equation in terms of the curvatures of a suitable q -deformation of the initial equation.

The rest of the paper is organized as follows. The first section is devoted to a review of the tannakian category of q -difference modules and to the curvature characterization of generic Galois groups established

²Notice that descent to algebraically closed field of constants have been performed in [Wib11] and [GGO11] (and in [DVH11c], using an idea of M. Wibmer). See Remark 2.5 below.

in [DVH11a]. In the second section, we introduce the differential tannakian framework and we show how it encompasses the Galois theory of Hardouin-Singer and the weak Picard-Vessiot ring of meromorphic solutions. In the third section, we state the comparison results between the differential Galois group of Hardouin-Singer and the generic, parametrized and algebraic, groups. The fourth section contains the results in the specialization of q , considered as a parameter.

1 The tannakian category of q -difference modules

Let K be a field and $K(x)$ be the field of rational functions in x with coefficients in K . The field $K(x)$ is naturally a q -difference algebra for any $q \in K \setminus \{0, 1, \text{roots of unity}\}$, *i.e.* is equipped with the operator

$$\begin{aligned} \sigma_q : K(x) &\longrightarrow K(x) \\ f(x) &\longmapsto f(qx) \end{aligned}.$$

More generally, we will consider a q -difference field (K, σ_q) and an extension \mathcal{F} of $K(x)$ equipped with a q -difference operator, still called σ_q , extending the action of σ_q .

Definition 1.1. A q -difference module over \mathcal{F} (of rank ν) is an \mathcal{F} -vector space $M_{\mathcal{F}}$ (of finite dimension ν) equipped with an invertible σ_q -semilinear operator, *i.e.*

$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } f \in \mathcal{F} \text{ and } m \in M_{\mathcal{F}}.$$

A morphism of q -difference modules over \mathcal{F} is a morphism of \mathcal{F} -vector spaces, commuting with the q -difference structures (for more generalities on the topic, *cf.* [vdPS97], [DV02, Part I] or [DVRSZ03]). We denote by $\text{Diff}(\mathcal{F}, \sigma_q)$ the category of q -difference modules over \mathcal{F} .

Let $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$ be a q -difference module over \mathcal{F} of rank ν . We fix a basis \underline{e} of $M_{\mathcal{F}}$ over \mathcal{F} and we set:

$$\Sigma_q \underline{e} = \underline{e}A,$$

with $A \in \text{GL}_{\nu}(\mathcal{F})$. A horizontal vector $\vec{y} \in \mathcal{F}^{\nu}$ with respect to the basis \underline{e} for the operator Σ_q is a vector that verifies $\Sigma_q(\underline{e}\vec{y}) = \underline{e}\vec{y}$, *i.e.* $\vec{y} = A\sigma_q(\vec{y})$. Therefore we call

$$\sigma_q(Y) = A^{-1}Y,$$

the (q -difference) system associated to $\mathcal{M}_{\mathcal{F}}$ (with respect to the basis \underline{e}).

It is well known that $\text{Diff}(\mathcal{F}, \sigma_q)$ is a tannakian category. It can be endowed with several fiber functors that give rise to different Galois groups (see the fundamental theorem of tannakian category [Del90, Theorem 7.1]). For an object $\mathcal{M} = (M, \sigma_q)$ of $\text{Diff}(\mathcal{F}, \sigma_q)$, we denote $\langle \mathcal{M} \rangle^{\otimes}$ the smallest full tannakian subcategory of $\text{Diff}(\mathcal{F}, \sigma_q)$ containing \mathcal{M} . For the reader convenience, we briefly recall the construction of some fiber functors for $\langle \mathcal{M} \rangle^{\otimes}$, already defined in the literature, before getting to the core of the present paper, in the next section.

1.1 Fiber functors associated with a Picard-Vessiot extensions

A Picard-Vessiot extension \mathcal{R} for an object \mathcal{M} of $\text{Diff}(\mathcal{F}, \sigma_q)$ is a simple σ_q -ring generated over \mathcal{F} by an invertible solution matrix of a q -difference system associated with some basis of \mathcal{M} . If the constants of \mathcal{R} coincide with the constant \mathcal{F}^{σ_q} of \mathcal{F} , then the Picard-Vessiot ring is neutral. It is proved in [And01, Theorem 3.4.2.3] that if the tannakian category $\langle \mathcal{M} \rangle^{\otimes}$ admits a fiber functor over \mathcal{F}^{σ_q} , we have an equivalence of quasi-inverse categories:

$$\{\text{fiber functor over } \mathcal{F}^{\sigma_q}\} \leftrightarrow \{\text{neutral Picard-Vessiot ring for } \mathcal{M}\}.$$

If the σ_q -constants \mathcal{F}^{σ_q} of \mathcal{F} are algebraically closed, a Picard-Vessiot ring \mathcal{R} always exists and is unique up to isomorphism, as proved in [vdPS97] and, if the characteristic of K is positive, in [vdPR07]. The fiber functor is given by

$$\begin{aligned} (1.1) \quad \omega : \quad \langle \mathcal{M} \rangle^{\otimes} &\longrightarrow \text{Vect}_{\mathcal{F}} \\ \mathcal{N} = (N, \Sigma_q) &\longmapsto \ker(\Sigma_q - \text{Id}, \mathcal{R} \otimes_{\mathcal{F}} N), \end{aligned}$$

where $Vect_{\mathcal{F}}$ is the category of finite dimensional \mathcal{F} -vector spaces. The group of \mathcal{F}^{σ_q} -points of $Aut^{\otimes}(\omega)$ coincides with the groups of automorphisms of \mathcal{R}/\mathcal{F} , that commute to σ_q .

1.2 Fiber functor associated with a meromorphic basis of solutions

For a fixed complex number q with $|q| \neq 1$, Praagman proves in [Pra86] that every linear q -difference equation with meromorphic coefficients over \mathbb{C}^* admits a basis of solutions, meromorphic over \mathbb{C}^* , linearly independent over the field of elliptic function C_E , *i.e.* the field of meromorphic functions over the elliptic curve $E := \mathbb{C}^*/q^{\mathbb{Z}}$. The reformulation of his theorem in the tannakian language is that the category of q -difference modules over the field of meromorphic functions on the punctured plane \mathbb{C}^* is a neutral tannakian category over C_E , *i.e.* admits a fiber functor into $Vect_{C_E}$. We give below the generic analogue of this theorem.

Let $K(x)$ be a q -difference field, $\partial = x \frac{d}{dx}$, $|\cdot|$ a norm on K such that $|q| > 1$ and C an algebraically closed field extension of K , complete w.r.t. $|\cdot|$.³ Here are a few examples to keep in mind:

- K is a subfield of \mathbb{C} equipped with the norm induced by \mathbb{C} and $C = \mathbb{C}$;
- K is finite extension of a field of rational functions $k(q)$, with no assumptions on the characteristic of k , equipped with the q^{-1} -adic norm;
- K is a finitely generated extension of \mathbb{Q} and q is an algebraic number, nor a root of unity: in this case there always exists a norm on the algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} in K such that $|q| > 1$, that can be extended to K . The field C is equal to \mathbb{C} if the norm is archimedean.

We call *holomorphic function over C^** a power series $f = \sum_{n=-\infty}^{\infty} a_n x^n$ with coefficients in C that satisfies

$$\lim_{n \rightarrow \infty} |a_n| \rho^n = 0 \text{ and } \lim_{n \rightarrow -\infty} |a_n| \rho^n = 0 \text{ for all } \rho > 0.$$

The holomorphic functions on C^* form a ring $\mathcal{H}ol(C^*)$. Its fraction field $\mathcal{M}er(C^*)$ is the field of meromorphic functions over C^* .

Proposition 1.2. *Every q -difference system $\sigma_q(Y) = AY$ with $A \in Gl_{\nu}(K(x))$ (and actually also $A \in Gl_{\nu}(\mathcal{M}er(C^*))$) admits a fundamental solution matrix with coefficients in $\mathcal{M}er(C^*)$, *i.e.* an invertible matrix $U \in Gl_{\nu}(\mathcal{M}er(C^*))$, such that $\sigma_q(U) = AU$.*

Remark 1.3. The proposition above is equivalent to the global triviality of the pull back over C^* of the fiber bundles on elliptic curves.

Proof. We are only sketching the proof. The Jacobi theta function

$$\Theta_q(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n,$$

is an element of $\mathcal{M}er(C^*)$. It is solution of the q -difference equation

$$y(qx) = qx y(x).$$

We follow [Sau00]. Since

- for any $c \in C^*$, the meromorphic function $\Theta(cx)/\Theta_q(x)$ is solution of $y(qx) = cy(x)$;
- the meromorphic function $x\Theta'_q(x)/\Theta_q(x)$ is solution of the equation $y(qx) = y(x) + 1$;

we can write a meromorphic fundamental solution to any regular singular system at 0, and, more generally, of any system whose Newton polygon has only one slope (*cf.* for instance [Sau00], [DVRSZ03] or [Sau04b, §1.2.2]). For the “pieces” of solutions linked to the Stokes phenomenon, all the technics of q -summation in the case $q \in \mathbb{C}$, $|q| > 1$, apply in a straightforward way to our situation (*cf.* [Sau04a, §2, §3]) and give a fundamental solution meromorphic over C^* . \square

³ What follows is of course valid also for the norms for which $|q| < 1$ and can be deduced by transforming the q -difference system $\sigma_q(Y) = AY$ in the q^{-1} -difference system $\sigma_{q^{-1}}(Y) = \sigma_{q^{-1}}(A^{-1})Y$.

The field of σ_q -constants of $\text{Mer}(C^*)$ is the field C_E of elliptic functions over the torus $E = C^*/q^{\mathbb{Z}}$. For any q -difference module \mathcal{M} over $K(x)$, there exists a weak Picard-Vessiot \mathcal{R}' ring which is generated over C_E by a fundamental meromorphic solution matrix. The adjective weak refers to the fact that \mathcal{R}' may not be a simple σ_q -ring (see [CHS08, Definition 2.1]). Nonetheless we obtain a C_E -linear fiber functor

$$(1.2) \quad \begin{aligned} \omega : \quad \langle \mathcal{M} \otimes_K C_E \rangle^{\otimes} &\longrightarrow \text{Vect}_{C_E} \\ \mathcal{N} = (N, \Sigma_q) &\longmapsto \ker(\Sigma_q - 1, \mathcal{R}' \otimes_{\mathcal{F}} N). \end{aligned}$$

Remark 1.4. The Picard-Vessiot group associated with the functor (1.1), for $\mathcal{F} = K(x)$, becomes isomorphic to the group associated with the fiber functor (1.2), after a convenient scalar extension (see [CHS08]).

In [Sau04b], Sauloy constructs a \mathbb{C} -linear fiber functor for q -difference modules over $\mathbb{C}(x)$, using a basis of meromorphic solutions. Since \mathbb{C} is algebraically closed, it follows from the classical general theory of tannakian categories, that such a fiber functor gives rise to a group that is isomorphic to the Picard-Vessiot group associated with (1.1), for $\mathcal{F} = \mathbb{C}(x)$. We won't consider Sauloy's point of view in this paper.

1.3 The forgetful functor and the generic Galois group

Following Katz's ideas [Kat82], we consider the generic Galois group studied in [DV02] and [DVH11a]. Although a more down to earth definition can be found in those references, such a group is associated with the forgetful fiber functor. Namely if \mathcal{M} is a q -difference module over \mathcal{F} , the forgetful functor

$$\begin{aligned} \eta_{\mathcal{F}} : \quad \langle \mathcal{M} \rangle^{\otimes} &\longrightarrow \text{Vect}_{\mathcal{F}} \\ \mathcal{N} = (N, \Sigma_q) &\longmapsto N. \end{aligned}$$

maps an object \mathcal{N} onto its underlying \mathcal{F} -vector space N . If $\mathcal{F} = K(x)$, it is possible to give an arithmetic description of the associated group $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) := \text{Aut}^{\otimes}(\eta)$, which we recall for further reference (see [DVH11a, §5] for more details). Let $\mathcal{M} = (M_{K(x)}, \Sigma_q)$ be a q -difference module over $K(x)$. We need the following notation:

- If the characteristic of K is zero and q is algebraic over \mathbb{Q} , but not a root of unity, we are in the following situation. We call Q the algebraic closure of \mathbb{Q} inside K , \mathcal{O}_Q the ring of integer of Q , \mathcal{C} the set of finite places v of Q and π_v a v -adic uniformizer. For almost all $v \in \mathcal{C}$ the following are well defined: the order κ_v , as a root of unity, of the reduction of q modulo π_v and the positive integer power ϕ_v of π_v , such that $\phi_v^{-1}(1 - q^{\kappa_v})$ is a unit of \mathcal{O}_Q . The field K has the form $Q(\underline{a}, b)$, where $\underline{a} = (a_1, \dots, a_r)$ is a transcendent basis of K/Q and b is a primitive element of the algebraic extension $K/Q(\underline{a})$. Choosing conveniently the set of generators \underline{a}, b and $P(x) \in \mathcal{O}_Q[\underline{a}, b, x]$, we can always find an algebra \mathcal{A} of the form

$$(1.3) \quad \mathcal{A} = \mathcal{O}_Q \left[\underline{a}, b, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots \right]$$

and a Σ_q -stable \mathcal{A} -lattice M of $\mathcal{M}_{K(x)}$, so that we can consider the $\mathcal{A}/(\phi_v)$ -linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

that we will call the v -curvature of $\mathcal{M}_{K(x)}$ -modulo ϕ_v . Notice that $\mathcal{O}_Q/(\phi_v)$ is not an integral domain in general.

- If q is transcendental over the fundamental field, \mathbb{Q} or \mathbb{F}_p , then there exists a subfield k of K such that K is a finite extension of $k(q)$. We denote by \mathcal{C} the set of places of K that extend the places of $k(q)$, associated to irreducible polynomials ϕ_v of $k[q]$, that vanish at roots of unity. Let κ_v be the order of the roots of ϕ_v , as roots of unity. Let \mathcal{O}_K be the integral closure of $k[q]$ in K . Choosing conveniently $P(x) \in \mathcal{O}_K[x]$, we can always find an algebra \mathcal{A} of the form:

$$(1.4) \quad \mathcal{A} = \mathcal{O}_K \left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots \right]$$

and a Σ_q -stable \mathcal{A} -lattice M of $\mathcal{M}_{K(x)}$, so that we can consider the $\mathcal{A}/(\phi_v)$ -linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

that we will call the *v-curvature of $\mathcal{M}_{K(x)}$ -modulo ϕ_v* . Notice that, once again, $\mathcal{O}_K/(\phi_v)$ is not an integral domain in general.

We recall that $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ is trivial if and only if there exists a basis \underline{e} of $M_{K(x)}$ over $K(x)$ such that $\Sigma_q \underline{e} = \underline{e}$. This is equivalent to ask that any q -difference systems associated to $\mathcal{M}_{K(x)}$ (with respect to any basis) has a fundamental solutions in $GL_{\nu}(K(x))$. Then the main result of [DVH11a] states:

Theorem 1.5 (cf. [DVH11a, Theorem 5.4]). *A q -difference module $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ over $K(x)$ is trivial if and only if there exists an algebra \mathcal{A} , as above, and a Σ_q -stable \mathcal{A} -lattice M of $M_{K(x)}$ such that the map*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

is the identity, for any v in a cofinite nonempty subset of \mathcal{C} .

The group $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is defined as the stabilizer of a line $L_{K(x)}$ in an object $\mathcal{W}_{K(x)} = (W_{K(x)}, \Sigma_q)$ of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes}$. The lattice M of $\mathcal{M}_{K(x)}$ determines a Σ_q -stable \mathcal{A} -lattice of all the objects of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$. In particular, the \mathcal{A} -lattice M of $M_{K(x)}$ determines an \mathcal{A} -lattice L of $L_{K(x)}$ and an \mathcal{A} -lattice W of $W_{K(x)}$.

Definition 1.6. Let $\tilde{\mathcal{C}}$ be a nonempty cofinite subset of \mathcal{C} and $(\Lambda_v)_{v \in \tilde{\mathcal{C}}}$ be a family of $\mathcal{A}/(\phi_v)$ -linear operators acting on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$. We say that *the algebraic group G contains the operators Λ_v modulo ϕ_v for almost all $v \in \mathcal{C}$* if for almost all (and at least one) $v \in \tilde{\mathcal{C}}$ the operator Λ_v stabilizes $L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ inside $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$:

$$\Lambda_v \in \text{Stab}_{\mathcal{A}/(\phi_v)}(L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)).$$

Theorem 1.5 is equivalent to the theorem below (see [DVH11a, Theorem 4.8]):

Theorem 1.7. *The generic group $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest closed algebraic subgroup of $\text{GL}(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v , for almost all $v \in \mathcal{C}$.*

The comparison theorems for the generic Galois group $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ are not present in the literature and they are proved below, at the same time as the comparison results for the parameterized generic Galois group (see §3).

2 The differential tannakian category of q -difference modules

From now on, we will assume that the characteristic of the base field is zero, with very few exceptions. The rule will be the following: if a statement concerns only objects whose definition does not involve derivations, it will hold also in the positive characteristic situations considered in the section above.

The aim of the tannakian formalism is to characterize the categories equivalent to a category of representations of a linear algebraic group which encodes the algebraic properties of the objects. Similarly, the aim of the differential tannakian formalism is to characterize the categories equivalent to a category of representations of a linear differential algebraic group which contains now a more precise information namely the differential algebraic relations satisfied by the objects. The differential tannakian formalism developed simultaneously by Ovchinnikov ([Ovc09a]) and Kamensky (see also [Kam]) is based on the axiomatic of the tannakian categories (see [Del90]) together with an additional endofunctor of the category, the prolongation. The prolongation of an object \mathcal{X} is a nontrivial extension of \mathcal{X} by itself and plays the role of a derivation in the category. We will refer to the work [Ovc09a] for the precise axiomatic and we describe below how we can endow the category $\text{Diff}(\mathcal{F}, \sigma_q)$ with a structure of differential tannakian category *i.e.* with a prolongation functor.

2.1 Differential tannakian structure of $\text{Diff}(\mathcal{F}, \sigma_q)$

From now on, let $(\mathcal{F}, \sigma_q, \partial)$ be a q -difference-differential field of zero characteristic, that is, an extension of $K(x)$ equipped with an extension of the q -difference operator σ_q and a derivation ∂ commuting with σ_q (cf. [Har08, §1.2]). For instance, the q -difference-differential field $(K(x), \sigma_q, x \frac{d}{dx})$ satisfies these assumptions. We can define an action of the derivation ∂ on the category $\text{Diff}(\mathcal{F}, \sigma_q)$, twisting the q -difference modules with the right \mathcal{F} -module $\mathcal{F}[\partial]_{\leq 1}$ of differential operators of order less or equal than one. We recall that the structure of right \mathcal{F} -module on $\mathcal{F}[\partial]_{\leq 1}$ is defined via the Leibniz rule, *i.e.*

$$\partial\lambda = \lambda\partial + \partial(\lambda), \text{ for any } \lambda \in \mathcal{F}.$$

Let V be an \mathcal{F} -vector space. We denote by $F_\partial(V)$ the tensor product of the right \mathcal{F} -module $\mathcal{F}[\partial]_{\leq 1}$ by the left \mathcal{F} -module V :

$$F_\partial(V) := \mathcal{F}[\partial]_{\leq 1} \otimes_{\mathcal{F}} V.$$

We will write simply v for $1 \otimes v \in F_\partial(V)$ and $\partial(v)$ for $\partial \otimes v \in F_\partial(V)$, so that $av + b\partial(v) := (a + b\partial) \otimes v$, for any $v \in V$ and $a + b\partial \in \mathcal{F}[\partial]_{\leq 1}$. Similarly to the constructions of [GM93, Proposition 16] for \mathcal{D} -modules, we endow $F_\partial(V)$ with a left \mathcal{F} -module structure such that if $\lambda \in \mathcal{F}$:

$$\lambda\partial(v) = \partial(\lambda v) - \partial(\lambda)v, \text{ for all } v \in V.$$

This construction comes out of the Leibniz rule $\partial(\lambda v) = \lambda\partial(v) + \partial(\lambda)v$, which justifies the notation introduced above.

Definition 2.1. The *prolongation functor* F_∂ is defined on the category of \mathcal{F} -vector spaces as follows. It associates to any object V the \mathcal{F} -vector space $F_\partial(V)$. If $f : V \rightarrow W$ is a morphism of \mathcal{F} -vector space then we define

$$F_\partial(f) : F_\partial(V) \rightarrow F_\partial(W),$$

setting $F_\partial(f)(\partial^k(v)) = \partial^k(f(v))$, for any $k = 0, 1$ and any $v \in V$.

The prolongation functor F_∂ restricts to a functor from the category $\text{Diff}(\mathcal{F}, \sigma_q)$ to itself in the following way:

1. If $\mathcal{M}_{\mathcal{F}} := (M_{\mathcal{F}}, \Sigma_q)$ is an object of $\text{Diff}(\mathcal{F}, \sigma_q)$ then $F_\partial(\mathcal{M}_{\mathcal{F}})$ is the q -difference module, whose underlying \mathcal{F} -vector space is $F_\partial(M_{\mathcal{F}}) = \mathcal{F}[\partial]_{\leq 1} \otimes M_{\mathcal{F}}$, as above, equipped with the q -invertible σ_q -semilinear operator defined by $\Sigma_q(\partial^k(m)) := \partial^k(\Sigma_q(m))$ for $k = 0, 1$.
2. If $f \in \text{Hom}(\mathcal{M}_{\mathcal{F}}, \mathcal{N}_{\mathcal{F}})$ then $F_\partial(f)$ is defined in the same way as for \mathcal{F} -vector spaces.

Remark 2.2. This formal definition comes from a simple and concrete idea (see [Har08]). Let $\mathcal{M}_{\mathcal{F}}$ be an object of $\text{Diff}(\mathcal{F}, \sigma_q)$. We fix a basis \underline{e} of $M_{\mathcal{F}}$ over \mathcal{F} such that $\Sigma_q \underline{e} = \underline{e}A$. Then $(\underline{e}, \partial(\underline{e}))$ is a basis of $F_\partial(M_{\mathcal{F}})$ and

$$\Sigma_q(\underline{e}, \partial(\underline{e})) = (\underline{e}, \partial(\underline{e})) \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix}.$$

In other terms, if $\sigma_q(Y) = A^{-1}Y$ is a q -difference system associated to $\mathcal{M}_{\mathcal{F}}$ with respect to a fixed basis \underline{e} , the q -difference system associated to $F_\partial(\mathcal{M}_{\mathcal{F}})$ with respect to the basis $\underline{e}, \partial(\underline{e})$ is:

$$\sigma_q(Z) = \begin{pmatrix} A^{-1} & \partial(A^{-1}) \\ 0 & A^{-1} \end{pmatrix} Z = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix}^{-1} Z.$$

If Y is a solution of $\sigma_q(Y) = A^{-1}Y$ in some q -difference-differential extension of \mathcal{F} then we have:

$$\sigma_q \begin{pmatrix} \partial Y \\ Y \end{pmatrix} = \begin{pmatrix} A^{-1} & \partial(A^{-1}) \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} \partial Y \\ Y \end{pmatrix},$$

in fact the commutation of σ_q and ∂ implies:

$$\sigma_q(\partial Y) = \partial(\sigma_q Y) = \partial(A^{-1} Y) = A^{-1} \partial Y + \partial(A^{-1}) Y.$$

Proposition 2.3. *The category $\text{Diff}(\mathcal{F}, \sigma_q)$, endowed with the prolongation functor F_∂ , is a differential tannakian category in the sense of [Ovc09a, Def.3].*

We are skipping the proof of this proposition, which is long but has no real difficulties. Let $\mathcal{M} \in \text{Diff}(\mathcal{F}, \sigma_q)$. We denote by $\langle \mathcal{M} \rangle^{\otimes, \partial}$ be the differential tannakian category generated by \mathcal{M} in $\text{Diff}(\mathcal{F}, \sigma_q)$.

Since we want to build an equivalence of category between $\text{Diff}(\mathcal{F}, \sigma_q)$ (or a differential tannakian subcategory \mathcal{C} of $\text{Diff}(\mathcal{F}, \sigma_q)$) with the category of differential representations of a linear differential algebraic group, we are interested with a special kind of fiber functors (*cf.* [Ovc09a, Def.4.1] for the general definition):

Definition 2.4. Let $\omega : \mathcal{C} \rightarrow \text{Vect}_{\mathcal{F}^{\sigma_q}}$ be a \mathcal{F}^{σ_q} -linear functor. We say that ω is a differential fiber functor for \mathcal{C} if

1. ω is a fiber functor in the sense of [SR72, 3.2.1.2];
2. $F_\partial \circ \omega = \omega \circ F_\partial$.

If $\text{Aut}^{\otimes, \partial}(\omega)$ denotes the group of differential tensor automorphism of ω ([Ovc09a, §4.3]), the category \mathcal{C} is equivalent to the category of differential representations of the linear differential algebraic group $\text{Aut}^{\otimes, \partial}(\omega)$. If $\mathcal{C} = \langle \mathcal{M} \rangle^{\otimes, \partial}$, for some $\mathcal{M} \in \text{Diff}(\mathcal{F}, \sigma_q)$, then we write $\text{Aut}^{\otimes, \partial}(\mathcal{M}, \omega)$ and $\text{Aut}^\otimes(\mathcal{M}, \omega)$ for the group of tensor automorphisms of the restriction of ω to the usual tannakian category $\langle \mathcal{M} \rangle^\otimes$.

If \mathcal{F}^{σ_q} is differentially closed (*cf.* [CS06, Sect.9.1] for definition and references), one can always construct a differential fiber functor (*cf.* [Ovc09a, Thm.16]) and two differential fiber functors are isomorphic. Notice that this is very much in the spirit of the tannakian formalism. In fact in [Del90, §7], P. Deligne proves that, if \mathcal{F}^{σ_q} is an algebraically closed field, the category $\text{Diff}(\mathcal{F}, \sigma_q)$ admits a fiber functor ω into the category $\text{Vect}_{\mathcal{F}^{\sigma_q}}$ of finite dimensional \mathcal{F}^{σ_q} -vector spaces.

To construct explicitly a differential fiber functor, we need to construct a fundamental solution matrix of a q -difference system associated to the q -difference module, with respect to some basis. The first approach is to make an abstract construction of an algebra containing a basis of abstract solutions of the q -difference module and all their derivatives (*cf.* [HS08, Definition 6.10]). We detail this approach in the next subsection. The major disadvantage of this construction is that it requires that the σ_q -constants of the base field form a differentially closed field, *i.e.* an enormous field. For this reason we will rather consider a differential fiber functor ω_E defined by meromorphic solutions of the module (*cf.* 2.3 below). In §2.4, we recall the definition of the parameterized generic Galois groups. Then, we will establish some comparison results between the parameterized generic Galois group, the group of differential tensor automorphism of ω_E and the Hardouin-Singer differential Galois group (*cf.* §3 below).

Remark 2.5. In [Wib11] and [GGO11] (and in [DVH11c], based on Wibmer's construction), one can find proofs of the fact that the Picard-Vessiot construction in [HS08] actually holds for an algebraically closed field of constants. The parameterized group considered in [HS08] is therefore defined over an algebraically closed field. We will continue to work with a differentially closed field of constants since it is more convenient in the proofs, although one should keep in mind that a descent is possible, at least on the algebraic closure of C_E .

2.2 Formal differential fiber functor

Let $(\mathcal{F}, \sigma_q, \partial)$ be a q -difference-differential field. In [HS08], the authors attach to a differential equation $\sigma_q(Y) = AY$ with $A \in \text{GL}_\nu(\mathcal{F})$, a (σ_q, ∂) -Picard-Vessiot ring: a simple (σ_q, ∂) -ring generated over \mathcal{F} by a fundamental solutions matrix of the system and all its derivatives w.r.t. ∂ . Here simple means with no non trivial ideal invariant under σ_q and ∂ (*cf.* [HS08, Definition 2.3]). Such (σ_q, ∂) -Picard-Vessiot rings always exist. A basic construction is to consider the ring of differential polynomials $S = \mathcal{F}\{Y, \frac{1}{\det Y}\}_{\partial}$, where Y is a matrix of differential indeterminates over \mathcal{F} of order ν , and to endow it with a q -difference operator compatible with the differential structure, *i.e.* such that

$$\sigma_q(Y) = AY, \sigma_q(\partial Y) = A\partial Y + \partial AY, \dots$$

Any quotient of the ring S by a maximal (σ_q, ∂) -ideal is a (σ_q, ∂) -Picard-Vessiot ring. If the σ_q -constants of a (σ_q, ∂) -Picard-Vessiot ring coincide with \mathcal{F}^{σ_q} , we say that this ring is neutral. The connection between neutral (σ_q, ∂) -Picard-Vessiot ring and differential fiber functor for \mathcal{M} is given by the following theorem which is the differential analogue of [And01, Theorem 3.4.2.3].

Theorem 2.6. *Let $\mathcal{M} \in \text{Diff}(\mathcal{F}, \sigma_q)$. If the differential tannakian category $\langle \mathcal{M} \rangle^{\otimes, \partial}$ admits a differential fiber functor over \mathcal{F}^{σ_q} , we have an equivalence of quasi-inverse categories:*

$$\{\text{differential fiber functor over } \mathcal{F}^{\sigma_q}\} \leftrightarrow \{\text{neutral } (\sigma_q, \partial) - \text{Picard-Vessiot ring}\}.$$

Proof. We only give a sketch of proof and refer to [Del90, Section 9] and to [And01, Theorem 3.4.2.3] for the algebraic proof. We consider the forgetful functor $\eta_{\mathcal{F}} : \langle \mathcal{M} \rangle^{\otimes, \partial} \mapsto \mathcal{F}$ -modules of finite type. If ω is a neutral differential fiber functor for $\langle \mathcal{M} \rangle^{\otimes, \partial}$, the functor $\text{Isom}^{\otimes, \partial}(\omega \otimes \mathbf{1}_{\mathcal{F}}, \eta_{\mathcal{F}})$ over the differential commutative \mathcal{F} -algebras, is representable by a differential \mathcal{F} -variety $\Sigma^{\partial}(\mathcal{M}, \omega)$. It is a $\text{Aut}^{\otimes, \partial}(\mathcal{M}, \omega)$ -torsor. The ring of regular functions $\mathcal{O}(\Sigma^{\partial}(\mathcal{M}, \omega))$, in the sense of Kolchin, of $\Sigma^{\partial}(\mathcal{M}, \omega)$, is a neutral (σ_q, ∂) -Picard-Vessiot extension for \mathcal{M} over \mathcal{F} . Conversely, let A be a neutral (σ_q, ∂) -Picard-Vessiot ring for \mathcal{M} . The functor $\omega_A : \langle \mathcal{M} \rangle^{\otimes, \partial} \mapsto \text{Vect}_{\mathcal{F}^{\sigma_q}}$ defined as follow, $\omega_A(\mathcal{N}) := \ker(\Sigma_q - \text{Id}, A \otimes \mathcal{N})$, is a neutral differential fiber functor. The functors $\omega \mapsto \mathcal{O}(\Sigma^{\partial}(\mathcal{M}, \omega))$ and $A \mapsto \omega_A$ are quasi-inverse. \square

As a corollary, we get that the differential tannakian category $\langle \mathcal{M} \rangle^{\otimes, \partial}$ admits a differential fiber functor over \mathcal{F}^{σ_q} if and only if there exists a neutral (σ_q, ∂) -Picard-Vessiot ring for \mathcal{M} . We recall below some consequences of Theorem 2.6.

Theorem 2.7. *Let $(\mathcal{F}, \sigma_q, \partial)$ be a q -difference-differential field. Let \mathcal{M} be an object of $\text{Diff}(\mathcal{F}, \sigma_q)$ and let R be a neutral (σ_q, ∂) -Picard-Vessiot ring for \mathcal{M} . Then,*

1. *the group of (σ_q, ∂) - \mathcal{F} -automorphisms G_R^{∂} of R coincides with the \mathcal{F}^{σ_q} -points of the linear differential algebraic group $\text{Aut}^{\otimes, \partial}(\mathcal{M}, \omega_R)$;*
2. *the differential dimension of $\text{Aut}^{\otimes, \partial}(\mathcal{M}, \omega_R)$ over \mathcal{F}^{σ_q} is equal to the differential transcendence degree of R over \mathcal{F} ;⁴*
3. *the linear differential algebraic group $\text{Aut}^{\otimes, \partial}(\mathcal{M}, \omega_R)$ is a Zariski dense subset in the linear algebraic group $\text{Aut}^{\otimes}(\mathcal{M}, \omega_R)$.*

Two neutral (σ_q, ∂) -Picard-Vessiot rings for \mathcal{M} become isomorphic over a differential closure of \mathcal{F}^{σ_q} . The same holds for two differential fiber functors.

Proof. See [Ovc09a] or [HS08, Prop. 6.18 and 6.26]. \square

As in the classical case, a sufficient condition to ensure the existence of a differential fiber functor or equivalently of a neutral (σ_q, ∂) -Picard-Vessiot, is that the field of σ_q -constants \mathcal{F}^{σ_q} is differentially closed. This assumption is very strong, since differentially closed field are enormous. We show in the next section, how, for q -difference equations over $K(x)$, one could weaken this assumption losing the simplicity of the Picard-Vessiot ring but requiring the neutrality (and considering in the process a ring of meromorphic solutions, rather than an abstract ring of solutions). We will speak, in that case, of *weak differential Picard-Vessiot ring* i.e. generated by the solutions and with no new constants. The corresponding algebraic notion was introduced in [CHS08, Definition 2.1].

2.3 Differential fiber functor associated with a basis of meromorphic solutions

We go back to the notation introduced in §1.1, i.e. we consider the q -difference-differential field $(C(x), \sigma_q, \partial = x \frac{d}{dx})$, where C is a complete algebraically closed normed extension of $(K, |\cdot|)$, with $|q| > 1$. Notice that both

⁴A (σ_q, ∂) -Picard-Vessiot ring R is a direct sum of copies of an integral domain S . By differential transcendence degree of R over \mathcal{F} , we mean the differential transcendence degree of the fraction field of S over \mathcal{F} .

$\mathcal{H}ol(C^*)$ and $\mathcal{M}er(C^*)$ are stable under the action of σ_q and ∂ . Because σ_q and ∂ commute, the derivation ∂ stabilizes C_E inside $\mathcal{M}er(C^*)$, so that C_E is naturally endowed with a structure of q -difference-differential field. Let \tilde{C}_E be a differential closure⁵ of C_E with respect to ∂ (cf. [CS06, §9.1]). We still denote by ∂ the derivation of \tilde{C}_E and we extend the action of σ_q to \tilde{C}_E by setting $\sigma_q|_{\tilde{C}_E} = id$. Let $C_E(x)$ (resp. $\tilde{C}_E(x)$) denote the field $C(x)(C_E)$ (resp. $C(x)(\tilde{C}_E)$)⁶.

We consider a q -difference module $\mathcal{M}_{K(x)}$ defined over $K(x)$ and the object $\mathcal{M}_{C_E(x)} := \mathcal{M}_{K(x)} \otimes_{K(x)} C_E(x)$ of $Diff(C_E(x), \sigma_q)$ obtained by scalar extension. Proposition 1.2 produces a fundamental matrix of solution $U \in Gl_\nu(\mathcal{M}er(C^*))$ of the q -difference system associated to $\mathcal{M}_{K(x)}$ with respect to a given basis \underline{e} of $\mathcal{M}_{K(x)}$ over $K(x)$. The (σ_q, ∂) -ring R_M generated over $C_E(x)$ by the entries of U and $1/\det(U)$ (cf. [HS08, Def.2.1]), i.e. the minimal q -difference-differential ring over $C_E(x)$ that contains U , $1/\det(U)$ and all its derivatives, is a subring of $\mathcal{M}er(C^*)$. It has the following properties:

Lemma 2.8. *The ring R_M is a (σ_q, ∂) -weak Picard-Vessiot ring for $\mathcal{M}_{C_E(x)}$ over $C_E(x)$, i.e. it is a (σ_q, ∂) -ring generated over $C_E(x)$ by a fundamental solutions matrix of the system associated to $\mathcal{M}_{C_E(x)}$, whose ring of σ_q -constants is equal to C_E . Moreover, it is an integral domain.*

Proof. Notice that $R_M \subset \mathcal{M}er(C^*)$ and that $C_E \subset R_M^{\sigma_q} \subset \mathcal{M}er(C^*)^{\sigma_q} = C_E$. \square

Let $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial}$ be the full differential tannakian subcategory generated by $\mathcal{M}_{C_E(x)}$ in $Diff(C_E(x), \sigma_q)$. For any object \mathcal{N} of $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial}$, we set

$$(2.1) \quad \omega_E(\mathcal{N}) := \ker(\Sigma_q - Id, R_M \otimes \mathcal{N})$$

Proposition 2.9. *The category $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial}$ equipped with the differential fiber functor*

$$\omega_E : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial} \rightarrow Vect_{C_E}$$

is a neutral differential tannakian category.

Proof. One has to check that the axioms of the definition in [Ovc09a] are verified. The verification is long but straightforward and the exact analogue of [CHS08, Proposition 3.6]. \square

Corollary 2.10. *The group of differential automorphisms $Aut^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ of ω_E is a linear differential algebraic group defined over C_E (cf. [Ovc09b, Def.8 and Thm.1]).*

Definition 2.11. We call $Aut^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ the differential Galois group of $\mathcal{M}_{C_E(x)}$.

Since R_M is not a (σ_q, ∂) -Picard-Vessiot ring, one can not conclude, as in Theorem 2.7, that the group of (σ_q, ∂) -automorphisms of R_M over $C_E(x)$ coincides with the group of C_E -points of $Aut^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ and that the differential dimension of $Aut^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ over C_E is equal to the differential transcendence degree of F_M , the fraction field of R_M over $C_E(x)$. We have to extend the scalars to the differential closure \tilde{C}_E of C_E in order to compare R_M with a (σ_q, ∂) -Picard-Vessiot ring of $\mathcal{M}_{\tilde{C}_E(x)}$ or, equivalently, ω_E with a differential fiber functor $\tilde{\omega}_E$ for $\mathcal{M}_{\tilde{C}_E(x)}$, which associates to any $\mathcal{N} \in \langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes, \partial}$:

$$(2.2) \quad \tilde{\omega}_E(\mathcal{N}) := \ker(\Sigma_q - Id, (R_M \otimes_{C_E} \tilde{C}_E) \otimes \mathcal{N}).$$

2.4 Parameterized generic Galois group

Let $(\mathcal{F}, \sigma_q, \partial)$ a q -difference-differential field as above. We denote by $\eta_{\mathcal{F}} : Diff(\mathcal{F}, \sigma_q) \rightarrow Vect_{\mathcal{F}}$, the forgetful functor from the category of q -difference modules over \mathcal{F} to the category of finite dimensional \mathcal{F} -vector spaces. The forgetful functor commutes with the prolongation functor F_{∂} :

$$F_{\partial} \circ \eta_{\mathcal{F}} = \eta_{\mathcal{F}} \circ F_{\partial},$$

⁵The differential closure of a field \mathcal{F} equipped with a derivation ∂ is a field $\tilde{\mathcal{F}}$ equipped with a derivation extending ∂ , with the property that any system of differential equations with coefficients in \mathcal{F} , having a solution in a differential extension of \mathcal{F} , has a solution in $\tilde{\mathcal{F}}$.

⁶ Notice that C_E (resp. \tilde{C}_E) and $C(x)$ are linearly disjoint over C . The field $\tilde{C}_E(x)$ is the generic analogue of the field $\mathcal{G}(x)$ in [HS08, p. 340].

Similarly to [Ovc08, Definition 8], we consider the functor $Aut^{\otimes, \partial}(\langle \mathcal{M} \rangle^{\otimes, \partial}, \eta_{\mathcal{F}})$ of differential tensor automorphism of the restriction of $\eta_{\mathcal{F}}$ to $\langle \mathcal{M} \rangle^{\otimes, \partial}$ defined over the category of \mathcal{F} -algebras. Then this functor is representable by a differential group (scheme) $Gal^{\partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ (for a more explicit description of such a group see [DVH11b]). The generic Galois group $Gal(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}})$ of \mathcal{M} introduced in §1.3 is the group of tensor automorphism of the restriction of $\eta_{\mathcal{F}}$ to $\langle \mathcal{M} \rangle^{\otimes}$.

As a motivation we anticipate the following consequence of the comparison results that we will show in §3 (more precisely cf. Corollary 3.9):

Corollary 2.12. *Let $\mathcal{M}_{K(x)}$ be a q -difference module defined over $K(x)$. Let $U \in Gl_{\nu}(Mer(C^*))$ be a fundamental solution matrix of $\mathcal{M}_{K(x)}$. Then, there exists a finitely generated extension K'/K such that the differential dimension of the differential field generated by the entries of U over $\tilde{C}_E(x)$ is equal to the differential dimension⁷ of $Gal^{\partial}(\mathcal{M}_{K(x)} \otimes_{K(x)} K'(x), \eta_{K'(x)})$.*

We recall that roughly speaking the ∂ -differential dimension of the total field of fraction F_M of R_M over $C_E(x)$ is equal to the maximal number of elements of F_M that are differentially independent over $C_E(x)$. So the differential dimension of $Gal^{\partial}(\mathcal{M}_{K(x)} \otimes_{K(x)} K'(x), \eta_{K'(x)})$ gives information on the number of meromorphic solutions of a q -difference equations that *do not* have any differential relation among them: it measures their hypertranscendence properties.

If $\mathcal{F} = K(x)$, where K is a finitely generated extension of \mathbb{Q} , we can give an arithmetic description of $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ (see [DVH11a, §5]). The last assumption on K is not restrictive for the sequel, since we will always be able to reduce to this case. We endow $K(x)$ with the derivation $\partial := x \frac{d}{dx}$, that commutes with σ_q . Let \mathcal{A} be the algebra constructed in 1.3. Notice that \mathcal{A} (in each case we have considered) is stable under the action of the derivation ∂ . Let $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$ be a q -difference module. The differential version of Chevalley's theorem (cf. [Cas72, Proposition 14], [MO10, Theorem 5.1]) implies that any closed differential subgroup G of $GL(M_{K(x)})$ can be defined as the stabilizer of some line $L_{K(x)}$ contained in an object $\mathcal{W}_{K(x)}$ of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$. Because the derivation does not modify the set of poles of a rational function, the lattice \mathcal{M} of $\mathcal{M}_{K(x)}$ determines a Σ_q -stable \mathcal{A} -lattice of all the objects of $\langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial}$. In particular, the \mathcal{A} -lattice M of $M_{K(x)}$ determines an \mathcal{A} -lattice L of $L_{K(x)}$ and an \mathcal{A} -lattice W of $W_{K(x)}$.

Definition 2.13. Let $\tilde{\mathcal{C}}$ be a nonempty cofinite subset of \mathcal{C} and $(\Lambda_v)_{v \in \tilde{\mathcal{C}}}$ be a family of $\mathcal{A}/(\phi_v)$ -linear operators acting on $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$. We say that *the differential group G contains the operators Λ_v modulo ϕ_v for almost all $v \in \mathcal{C}$* if for almost all (and at least one) $v \in \tilde{\mathcal{C}}$ the operator Λ_v stabilizes $L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ inside $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$:

$$\Lambda_v \in Stab_{\mathcal{A}/(\phi_v)}(L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)).$$

Theorem 1.5 is equivalent to the following statement:

Theorem 2.14 ((See [DVH11b, Theorem 2.4]). *The differential group $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest closed differential subgroup of $GL(M_{K(x)})$ that contains the operators $\Sigma_q^{\kappa_v}$ modulo ϕ_v , for almost all $v \in \mathcal{C}$.*

3 Comparison of Galois groups

Let K be a field and $|\cdot|$ a norm on K such that $|q| > 1$. We will be dealing with groups defined over the following fields:

C = smallest algebraically closed and complete extension of the normed field $(K, |\cdot|)$;

C_E = field of constants with respect to σ_q of $Mer(C^*)$;

\overline{C}_E = algebraic closure of C_E ;

\tilde{C}_E = differential closure of C_E .

We remind that any q -difference system $Y(qx) = A(x)Y(x)$, with $A(x) \in Gl_{\nu}(K(x))$ has a fundamental solution in $Mer(C^*)$ (cf. Proposition 1.2).

⁷cf. [HS08, p. 337] for definition and references.

Let $\mathcal{M}_{K(x)}$ be a q -difference module over $K(x)$. For any q -difference field extension $\mathcal{F}/K(x)$ we will denote by $\mathcal{M}_{\mathcal{F}}$ the q -difference module over \mathcal{F} obtained from $\mathcal{M}_{K(x)}$ by scalar extension. We can attach to $\mathcal{M}_{K(x)}$ a collection of fiber and differential fiber functors defined upon the above field extensions. As explained in Theorem 2.7, the groups of tensor or differential tensor automorphisms attached to these neutral functors correspond to classical notions of Galois groups of a q -difference equation, what we call here the Picard-Vessiot groups. Their definition rely on adapted notion of admissible solutions and their dimension measure the algebraic and differential, when it make sense, behavior of these solutions. We give a precise description of some of these Picard-Vessiot groups below.

In [vdPS97, §1.1], Singer and van der Put attached to the q -difference module $\mathcal{M}_{C(x)} := \mathcal{M}_{K(x)} \otimes C(x)$ a Picard-Vessiot ring R which is a q -difference extension of $C(x)$, containing abstract solutions of the module. This means that the q -difference module $\mathcal{M}_{C(x)} \otimes R$ is trivial. Therefore, the functor ω_C from the subcategory $\langle \mathcal{M}_{C(x)} \rangle^{\otimes}$ of $\text{Diff}(C(x), \sigma_q)$ into Vect_C defined by

$$\omega_C(\mathcal{N}) := \ker(\Sigma_q - \text{Id}, R \otimes_{C(x)} \mathcal{N})$$

is a fiber functor. Since $R \otimes_C C_E$ is a *weak* Picard-Vessiot ring (cf. [CHS08, Def.2.1]), we can also introduce the functor ω_{C_E} from the subcategory $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes}$ of $\text{Diff}(C_E(x), \sigma_q)$ into Vect_{C_E} :

$$\omega_{C_E}(\mathcal{N}) := \ker(\Sigma_q - \text{Id}, (R \otimes_C C_E) \otimes_{C_E(x)} \mathcal{N}).$$

One can prove that ω_{C_E} is actually a fiber functor (cf. [CHS08, Prop.3.6]). We have therefore many different fiber functors. First of all the functors defined above:

$$(3.1) \quad \omega_C : \langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_C, \quad \mathcal{N} \mapsto \ker(\Sigma_q - \text{Id}, R \otimes_{C(x)} \mathcal{N});$$

$$(3.2) \quad \omega_{C_E} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{C_E}, \quad \mathcal{N} \mapsto \ker(\Sigma_q - \text{Id}, (R \otimes_C C_E) \otimes_{C_E(x)} \mathcal{N});$$

$$(3.3) \quad \omega_{\tilde{C}_E} : \langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{\tilde{C}_E}, \quad \mathcal{N} \mapsto \ker(\Sigma_q - \text{Id}, (R \otimes_C \tilde{C}_E) \otimes_{\tilde{C}_E(x)} \mathcal{N});$$

Then we have the fiber functors associated with the meromorphic basis of solutions⁸:

$$(3.4) \quad \omega_E : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{C_E} \quad (\text{which is the restriction of (2.1)});$$

$$(3.5) \quad \tilde{\omega}_E : \langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{\tilde{C}_E} \quad (\text{which is the restriction of (2.2)});$$

two differential fiber functors extending the fiber functors with the same name above:

$$(3.6) \quad \omega_E : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{C_E};$$

$$(3.7) \quad \tilde{\omega}_E : \langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\tilde{C}_E}.$$

Notice that $\omega_{\tilde{C}_E}$ and $\tilde{\omega}_E$ are isomorphic as fiber functors over $\langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes}$, since \tilde{C}_E is algebraically closed. Moreover two differential fiber functor over \tilde{C}_E are isomorphic so that $\tilde{\omega}_E$ could actually be any \tilde{C}_E -linear fiber functor.

Finally we have four forgetful functors:

$$(3.8) \quad \eta_{K(x)} : \langle \mathcal{M}_{K(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{K(x)} \quad \text{and its extension to } \langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial};$$

$$(3.9) \quad \eta_{C(x)} : \langle \mathcal{M}_{C(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{C(x)} \quad \text{and its extension to } \langle \mathcal{M}_{C(x)} \rangle^{\otimes, \partial};$$

$$(3.10) \quad \eta_{C_E(x)} : \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{C_E(x)} \quad \text{and its extension to } \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial};$$

$$(3.11) \quad \eta_{\tilde{C}_E(x)} : \langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes} \longrightarrow \text{Vect}_{\tilde{C}_E} \quad \text{and its extension to } \langle \mathcal{M}_{\tilde{C}_E(x)} \rangle^{\otimes, \partial}.$$

The group of tensor automorphisms of ω_C corresponds to the “classical” Picard-Vessiot group of a q -difference equation attached to $\mathcal{M}_{K(x)}$, defined in [vdPS97, §1.2]. It can be identified to the group of ring

⁸Notice that $\omega_E(\mathcal{N}) = \ker(\Sigma_q - \text{Id}, R_M \otimes \mathcal{N})$, where $R_M = C_E(x)\{U, \det U^{-1}\}$ is the smallest ∂ -ring containing $C_E(x)$, the entries of U and $\det U^{-1}$. To define ω_E over $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes}$ we should have considered the classical Picard-Vessiot extension $C_E(x)[U, \det U^{-1}]$. Anyway, since \mathcal{M} is trivialized both on R_M and $C_E(x)[U, \det U^{-1}]$ and $R_M^{\sigma_q} = C_E(x)[U, \det U^{-1}]^{\sigma_q} = C_E$, the q -analogue of the wronskian lemma implies that $\ker(\Sigma_q - \text{Id}, R_M \otimes \mathcal{M}) = \ker(\Sigma_q - \text{Id}, C_E(x)[U, \det U^{-1}] \otimes \mathcal{M})$, as C_E -vector spaces. The same holds for any object of the category $\langle \mathcal{M}_{C_E(x)} \rangle^{\otimes}$.

automorphisms of R stabilizing $C(x)$ and commuting with σ_q . Its dimension as a linear algebraic group is equal to the “transcendence degree” of the total ring of quotients of R over $C(x)$, *i.e.* it measures the algebraic relations between the formal solutions introduced by Singer and van der Put over $C(x)$.

The group of tensor automorphisms of ω_E corresponds to another Picard-Vessiot group attached to $\mathcal{M}_{K(x)}$. Its dimension as a linear algebraic group is equal to the transcendence degree of the fraction field F_M of R_M over $C_E(x)$. In other words, $\text{Aut}^\otimes(\mathcal{M}_{C_E(x)}, \omega_E)$ measures the algebraic relations between the meromorphic solutions, we have introduced in §2.3. One of the main results of [CHS08] is

Theorem 3.1. *The linear algebraic groups $\text{Aut}^\otimes(\mathcal{M}_{C(x)}, \omega_C)$, $\text{Aut}^\otimes(\mathcal{M}_{C_E(x)}, \omega_{C_E})$, $\text{Aut}^\otimes(\mathcal{M}_{C_E(x)}, \omega_E)$ and $\text{Aut}^\otimes(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ become isomorphic over \tilde{C}_E .*

The goal of the next sections is to relate the generic (parametrized and algebraic) Galois group of $\mathcal{M}_{K(x)}$ with the Picard-Vessiot groups and thus with the algebraic and differential behavior of the meromorphic solutions of $\mathcal{M}_{K(x)}$. In a first place, we prove a differential analogue of Theorem 3.1. To conclude, we show how the curvature criteria lead to the comparison between the parametrized generic Galois group over $C(x)$ and the differential tannakian group induced by $\tilde{\omega}_E$.

3.1 Differential Picard-Vessiot groups over the elliptic functions

In this section, we adapt the technics of [CHS08, Section 2] to a differential framework, in order to compare the distinct (σ_q, ∂) -Picard-Vessiot rings, neutral and weak, attached to $\mathcal{M}_{K(x)}$ over C_E and \tilde{C}_E . For a model theoretic approach of these questions, we refer to [PN09].

Let $R_M = C_E(x)\{U, \frac{1}{\det U}\}_\partial$ be the weak (hence not simple) (σ_q, ∂) -Picard-Vessiot ring attached to $\mathcal{M}_{C_E(x)}$, with $U \in \text{Gl}_\nu(\text{Mer}(C^*))$ a fundamental solutions matrix of $\sigma_q(Y) = AY$, a q -difference system attached to $\mathcal{M}_{K(x)}$ with $A \in \text{Gl}_\nu(K(x))$. The ring R_M allows to define the fiber functor ω_E , as seen previously. It follows from Theorem 2.6, there exists a neutral (simple) (σ_q, ∂) -Picard-Vessiot ring R'_M which also defines ω_E . Adapting to a differential context [CHS08, Proposition 2.7], we can compare R_M and R'_M :

Proposition 3.2. *Let $F_M = C_E(x)\langle U \rangle_\partial$ be the fraction field of R_M , *i.e.* the field extension of $C_E(x)$ differentially generated by U . There exists a (σ_q, ∂) - $C_E(x)$ -embedding $\rho : R'_M \rightarrow F_M \otimes \tilde{C}_E$, where σ_q acts on $F_M \otimes \tilde{C}_E$ via $\sigma_q(f \otimes c) = \sigma_q(f) \otimes c$.*

Proof. Let $Y = (Y_{(i,j)})$ be a $\nu \times \nu$ -matrix of differential indeterminates over F_M . We have $S = C_E(x)\{Y, \frac{1}{\det Y}\}_\partial \subset F_M\{Y, \frac{1}{\det Y}\}_\partial$. As in §2.2, we endow $F_M\{Y, \frac{1}{\det Y}\}_\partial$ with a q -difference structure compatible with the differential structure by setting $\sigma_q(Y) = AY$. One may assume that $R'_M = S/\mathfrak{M}$ where \mathfrak{M} be a maximal (σ_q, ∂) -ideal of S . Put $X = U^{-1}Y$ in $F_M\{Y, \frac{1}{\det Y}\}_\partial$. One has $\sigma_q(X) = X$ and $F_M\{Y, \frac{1}{\det Y}\}_\partial = F_M\{X, \frac{1}{\det X}\}_\partial$. Let $S' = C_E\{X, \frac{1}{\det X}\}_\partial$. The ideal \mathfrak{M} generates a proper (σ_q, ∂) -ideal (\mathfrak{M}) in $F_M\{Y, \frac{1}{\det Y}\}_\partial$. By [HS08, Lemma 6.12], the map $I \mapsto I \cap S'$ induces a bijective correspondence from the set of (σ_q, ∂) -ideals of $F_M\{X, \frac{1}{\det X}\}_\partial$ and the set of ∂ -ideals of $C_E\{X, \frac{1}{\det X}\}_\partial$. We let $\tilde{\mathfrak{M}} = (\mathfrak{M}) \cap S'$ and \mathfrak{P} is a maximal differential ideal of S' containing $\tilde{\mathfrak{M}}$. The differential ring S'/\mathfrak{P} is an integral domain and its fraction field is a finitely generated constrained extension of C_E (*cf.* [Kol74, p.143]). By [Kol74, Corollary 3], there exists a differential homomorphism $S'/\mathfrak{P} \rightarrow \tilde{C}_E$. We then have

$$S' \rightarrow S'/\mathfrak{P} \rightarrow \tilde{C}_E.$$

One can extend this differential homomorphism into a (σ_q, ∂) -homomorphism

$$\phi : F_M \left\{ Y, \frac{1}{\det Y} \right\}_\partial = F_M \otimes_{C_E} S' \rightarrow F_M \otimes_{C_E} \tilde{C}_E.$$

The kernel of the restriction of ϕ to S contains \mathfrak{M} . Since \mathfrak{M} is a maximal (σ_q, ∂) -ideal, this kernel is equal to \mathfrak{M} . Then, ϕ induces an embedding $R'_M \rightarrow F_M \otimes_{C_E} \tilde{C}_E$. \square

As in Theorem 2.7, let $G_{R_M}^\partial$ (resp. $G_{R'_M}^\partial$) be the group of (σ_q, ∂) - $C_E(x)$ -automorphisms of R_M (resp. R'_M). The group $G_{R'_M}^\partial$ consists in the C_E -points of $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ and similarly to [CHS08, Proposition 2.2], one can prove that $G_{R_M}^\partial$ corresponds to the C_E -points of a linear differential algebraic group defined over C_E whose differential dimension equals the transcendence degree of F_M over $C_E(x)$. We have

Corollary 3.3. *Let R_M, F_M, R'_M be as above. The morphism ρ maps $R'_M \otimes_{C_E} \tilde{C}_E$ isomorphically on $R_M \otimes_{C_E} \tilde{C}_E$. The isomorphism ρ induces an isomorphism of differential algebraic groups between $G_{R'_M}^\partial$ and $G_{R_M}^\partial$ over \tilde{C}_E .*

Proof. Let $V \in \text{Gl}_\nu(R'_M)$ a fundamental solution matrix such that $R'_M \otimes_{C_E(x)} \tilde{C}_E(x) = \tilde{C}_E(x) \{V, \frac{1}{\det V}\} \partial$ and $U \in \text{Gl}_\nu(R_M)$ as in the proof above. Then $\rho(V) = U.C$ where $C \in \text{Gl}_\nu(\tilde{C}_E)$. Therefore ρ is an isomorphism. A differential analogue of [CHS08, Corollary 2.5] combined with the isomorphism ρ yields to the announced group isomorphism. \square

This corollary shows that a weak Picard-Vessiot ring, here R_M , becomes isomorphic to a neutral Picard-Vessiot ring, here R'_M over a differentially closed field extension. The same holds for their groups of (σ_q, ∂) -automorphism. Now, it remains to compare the neutral (σ_q, ∂) -Picard-Vessiot ring R'_M to a (σ_q, ∂) -Picard-Vessiot ring attached to a neutral differential fiber functor $\tilde{\omega}_E$ over \tilde{C}_E . A differential analogue of [CHS08, Proposition 2.4 and Corollary 2.5] gives

Proposition 3.4. *The ring $\tilde{R}_M := R'_M \otimes_{C_E(x)} \tilde{C}_E(x)$ is a (σ_q, ∂) -Picard-Vessiot ring for $\mathcal{M}_{\tilde{C}_E(x)}$. As in Theorem 2.7, let $G_{\tilde{R}_M}^\partial$ be the group (σ_q, ∂) -automorphisms of \tilde{R}_M over $\tilde{C}_E(x)$. The linear differential algebraic groups $G_{R'_M}^\partial$ and $G_{\tilde{R}_M}^\partial$ are isomorphic over \tilde{C}_E .*

Combining the previous results and some generalities about neutral differential fiber functors (cf. Theorem 2.7), we find

Theorem 3.5. *In the notation introduced above we have:*

1. *the linear differential algebraic group $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ corresponds to the differential Galois group attached to $\mathcal{M}_{\tilde{C}_E(x)}$ via $\omega_{\tilde{C}_E}$ (i.e. the group constructed in [HS08, Theorem 2.6]) and is isomorphic over \tilde{C}_E to $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$;*
2. *the differential transcendence degree of the differential field generated over $\tilde{C}_E(x)$ by a basis of meromorphic solutions of $\mathcal{M}_{K(x)}$ is equal to the differential dimension of $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ over \tilde{C}_E .*

Proof. By Theorem 2.7. 1), the linear algebraic group $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$ (resp. $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$) corresponds to the differential Galois group $G_{\tilde{R}_M}^\partial$ of Hardouin-Singer (resp. to the automorphism group $G_{R'_M}^\partial$ of the neutral Picard-vessiot ring R'_M). Proposition 3.4 combined with Corollary 3.3 yields to the required isomorphism. By Theorem 2.7. 2), the differential dimension of $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$ is equal to the differential transcendence degree of R'_M over C_E . The isomorphism between R'_M and R_M over \tilde{C}_E ends the proof. \square

Remark 3.6. The results of this section are still valid for any q -difference module \mathcal{M} over $K(x)$ with R_M any integral weak (σ_q, ∂) -Picard-Vessiot ring and \tilde{R}_M a (σ_q, ∂) -Picard-Vessiot ring for $\mathcal{M} \otimes_{K(x)} K(\tilde{C}_K)$ where \tilde{C}_K is a differential closure of the σ_q -constants of K .

3.2 Comparison results for generic Galois groups

We are now concerned with the generic Galois groups, algebraic and parametrized. We first relate them with the Picard-Vessiot groups we have studied previously and then we investigate how they behave through certain type of base field extensions. The warning at the beginning of §2 still holds.

3.2.1 Comparison with Picard-Vessiot groups

Let $\mathcal{M}_{K(x)}$ be a q -difference module defined over $K(x)$. We have attached to $\mathcal{M}_{K(x)}$ the following groups:

group	fiber functor	field of definition
$Aut^\otimes(\mathcal{M}_{C(x)}, \omega_C)$	$\omega_C: \langle \mathcal{M}_{C(x)} \rangle^\otimes \longrightarrow Vect_C$	C
$Gal(\mathcal{M}_{C(x)}, \eta_{C(x)})$	$\eta_{C(x)}: \langle \mathcal{M}_{C(x)} \rangle^\otimes \longrightarrow Vect_{C(x)}$	$C(x)$
$Gal^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)})$	$\eta_{C(x)}: \langle \mathcal{M}_{C(x)} \rangle^{\otimes, \partial} \longrightarrow Vect_{C(x)}$	$C(x)$
$Aut^\otimes(\mathcal{M}_{C_E(x)}, \omega_E)$	$\omega_E: \langle \mathcal{M}_{C_E(x)} \rangle^\otimes \longrightarrow Vect_{C_E}$	C_E
$Aut^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E)$	$\omega_E: \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial} \longrightarrow Vect_{C_E}$	C_E
$Gal(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$	$\eta_{C_E(x)}: \langle \mathcal{M}_{C_E(x)} \rangle^\otimes \longrightarrow Vect_{C_E(x)}$	$C_E(x)$
$Gal^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$	$\eta_{C_E(x)}: \langle \mathcal{M}_{C_E(x)} \rangle^{\otimes, \partial} \longrightarrow Vect_{C_E(x)}$	$C_E(x)$

In order to relate the generic Galois groups and the groups defined by tensor automorphisms of fiber functors, we need to investigate first the structure of the different Picard-Vessiot rings, one can attach to \mathcal{M} . So first, let R be the Picard-Vessiot ring over $C(x)$, defined by Singer and van der Put. In general, R is a sum of domains $R = R_0 \oplus \dots \oplus R_{t-1}$, where each component R_i is invariant under the action of σ_q^t . The positive integer t corresponds to the number of connected components of the q -difference Galois group $Aut^\otimes(\mathcal{M}_{C(x)}, \omega_C)$ of $\mathcal{M}_{C(x)}$. Following [vdPS97, Lemma 1.26], we consider now $\mathcal{M}_{C(x)}^t$, the t -th iterate of $\mathcal{M}_{C(x)}$, which is a q^t -difference module over $C(x)$. Since the Picard-Vessiot ring of $\mathcal{M}_{C(x)}^t$ is isomorphic to one of the components of R , say R_0 , its q^t -difference Galois group (resp. its generic Galois group) is equal to the identity component of $Aut^\otimes(\mathcal{M}_{C(x)}, \omega_C)$ (resp. $Gal(\mathcal{M}_{C(x)}, \eta_C)$). Then, let R_M be the weak Picard-Vessiot ring attached to $\mathcal{M}_{C_E(x)}$ over $C_E(x)$, already considered above. It is an integral domain and its subfield of constants is C_E (see Lemma 2.8).

Proposition 3.7. *Let us denote by F_0 and F_M the fractions fields of R_0 and R_M . We have the following isomorphisms of linear algebraic groups:*

1. $Aut^\otimes(\mathcal{M}_{C(x)}, \omega_C)^\circ \otimes_K F_0 \simeq Gal(\mathcal{M}, \eta_{C(x)})^\circ \otimes_{K(x)} F_0$, where G° denotes the identity component of a group G ;
2. $Aut^\otimes(\mathcal{M}_{C_E(x)}, \omega_E) \otimes_{C_E} F_M \simeq Gal(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} F_M$;

and also the isomorphisms of linear differential algebraic groups:

3. $Aut^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E) \otimes_{C_E} F_M \simeq Gal^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} F_M$.

Proof. This is an analogue of [Kat82, Proposition 4.1] and we only give a sketch of proof in the case of $\mathcal{M}_{C_E(x)}$. Since R_M is a (σ_q, ∂) -Picard-Vessiot ring, we have an isomorphism of R_M -module between

$$\omega_E(\mathcal{M}_{C_E(x)}) \otimes_{C_E} R_M = \ker(\Sigma_q - Id, R_M \otimes \mathcal{M}_{C_E(x)}) \otimes R_M \simeq \mathcal{M}_{C_E(x)} \otimes_{C_E(x)} R_M.$$

Extending the scalars from R_M to F_M yields to the required isomorphism

$$\omega_E(\mathcal{M}_{C_E(x)}) \otimes_{C_E} F_M \simeq \mathcal{M}_{C_E(x)} \otimes_{C_E(x)} F_M,$$

which in view of its construction is compatible with the constructions of differential linear algebra. In particular, if $\mathcal{W} \subset Constr_{C_E(x)}^\partial(\mathcal{M}_{C_E(x)})$ then we have,

$$\omega_E(\mathcal{W}) \otimes_{C_E} F_M \simeq \mathcal{W} \otimes_{C_E(x)} F_M,$$

inside $\text{Constr}_{C_E}^\partial(\omega_E(\mathcal{M}_{C_E(x)})) \otimes_{C_E} F_M \simeq \text{Constr}_{C_E(x)}^\partial(\mathcal{M}_{C_E(x)}) \otimes_{C_E(x)} F_M$. These canonical identifications give a canonical isomorphism of linear differential algebraic groups over F_M ,

$$\text{Aut}^{\otimes, \partial}(\mathcal{M}_{C_E(x)}, \omega_E) \otimes_{C_E} F_M \simeq \text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \otimes_{C_E(x)} F_M.$$

This ends the proof. \square

Remark 3.8. This proposition expresses the fact that the Picard-Vessiot ring is a bitorsor (differential bitorsor when it makes sense) under the action of the generic (parametrized) Galois group and the Picard-Vessiot (parametrized) group.

Since the dimension of a differential algebraic group as well as the differential transcendence degree of a field extension do not vary up to field extension one has proved the following corollary

Corollary 3.9. *Let $\mathcal{M}_{K(x)}$ be a q -difference module defined over $K(x)$. Then, the ∂ -differential dimension of F_M (cf. [HS08, p. 337] for definition and references) over $C_E(x)$ is equal to the differential dimension of $\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$.*

Proof. Theorem 3.5 and Proposition 3.7 give the desired equality. \square

3.2.2 Reduction to $C(x)$ and $C_E(x)$

The following lemma shows how, for any field extension L/K , the parametrized generic Galois group of $\mathcal{M}_{L(x)}$ is equal, up to scalar extension, to the parametrized generic Galois group of $\mathcal{M}_{K'(x)}$, for a convenient finitely generated extension K'/K , with $K' \subset L$.

Lemma 3.10. *Let L be a field extension of K with $\sigma_q|_L = \text{id}$. There exists a finitely generated intermediate field $L/K'/K$ such that*

$$\text{Gal}(\mathcal{M}_{L(x)}, \eta_{L(x)}) \cong \text{Gal}(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes_{K'(x)} L(x)$$

and

$$\text{Gal}^\partial(\mathcal{M}_{L(x)}, \eta_{L(x)}) \cong \text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes_{K'} L(x).$$

These equalities hold when we replace K' by any subfield extension of L containing K' .

Proof. By definition, $\text{Gal}^\partial(\mathcal{M}_{L(x)}, \eta_{L(x)})$ is the stabilizer inside $\text{Gl}(M_{L(x)})$ of all $L(x)$ -vector spaces of the form $W_{L(x)}$ for W object of $\langle \mathcal{M}_{L(x)} \rangle^{\otimes, \partial}$. Similarly, for any field extension $L/K'/K$, we have

$$\text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) = \text{Stab}(W_{K'(x)}, \mathcal{W} \text{ object of } \langle \mathcal{M}_{K'(x)} \rangle^{\otimes, \partial}).$$

Then,

$$\text{Gal}^\partial(\mathcal{M}_{L(x)}, \eta_{L(x)}) \subset \text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes L(x).$$

By noetherianity, the (parametrized) generic Galois group of $\mathcal{M}_{L(x)}$ is defined by a finite family of (differential) polynomial equations, thus we can choose K' more carefully. \square

Since K'/K is of finite type, if we can calculate the group $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ (resp. $\text{Gal}^\partial(\mathcal{M}_{K(x)}, \eta_{K(x)})$) by a curvature procedure, the same holds for the group $\text{Gal}(\mathcal{M}_{K'(x)}, \eta_{K'(x)})$ (resp. $\text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)})$) and thus for $\text{Gal}(\mathcal{M}_{L(x)}, \eta_{L(x)})$ (resp. $\text{Gal}^\partial(\mathcal{M}_{L(x)}, \eta_{L(x)})$). Applying these considerations to $L = C$ or $L = C_E$, we will forget the field K for a while, keeping in mind that the generic Galois group of \mathcal{M} over $C(x)$ or over $C_E(x)$, may also be computed with the help of curvatures defined over a smaller field.

Proposition 3.11. *The differential linear algebraic group $\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$ is defined over $C(x)$ and we have isomorphism of linear differential algebraic groups:*

$$\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \xrightarrow{\sim} \text{Gal}^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)}) \otimes_{C_E(x)} C_E(x).$$

The same holds for the generic Galois groups, i.e. we have an isomorphism of linear algebraic groups

$$\text{Gal}(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \xrightarrow{\sim} \text{Gal}(\mathcal{M}_{C(x)}, \eta_{C(x)}) \otimes_{C_E(x)} C_E(x).$$

Proof. We give the proof in the differential case. The same argument than in the proof of Lemma 3.10 gives the inclusion

$$\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \subset \text{Gal}^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)}) \otimes C_E(x).$$

The group $\text{Aut}^\partial(C_E/C)$ of C -differential automorphism of C_E acts on $\mathcal{M}_{C_E(x)} = \mathcal{M}_{C(x)} \otimes C_E$ via the semi-linear action $(\tau \rightarrow id \otimes \tau)$. Thus the latter group acts on $\text{Constr}_{C_E(x)}^\partial(\mathcal{M}_{C_E(x)}) = \text{Constr}_{C(x)}^\partial(\mathcal{M}_{C(x)}) \otimes C_E$. Since this action commutes with σ_q , it therefore permutes the subobjects of $\text{Diff}(C_E(x), \sigma_q)$ contained in $\mathcal{M}_{C_E(x)}$. Since $C_E(x)^{\text{Aut}^\partial(C_E/C)} = C(x)$ (cf. [CHS08, Lemma 3.3]), we obtain that $\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$ is defined over $C(x)$. Putting all together, we have shown that $\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)})$ is equal to $G \otimes_{C(x)} C_E(x)$ where G is a linear differential subgroup of $\text{Gal}^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)})$ defined over $C(x)$. This implies that we can choose a line L in a construction of differential algebra of $\mathcal{M}_{C(x)}$ such that $G = \text{Stab}(L)$. By Lemma 3.10, there exists a finitely generated extension K'/K , such that $K' \subset C$ and that:

- $\text{Gal}^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)}) \cong \text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes C(x)$;
- the line L is defined over $K'(x)$ (and hence so does G).

Since C_E is purely transcendental over the algebraically closed field C , we call also choose a purely transcendental finitely generated extension K''/K' , with $K'' \subset C_E$, such that

$$\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) \cong \text{Gal}^\partial(\mathcal{M}_{K''(x)}, \eta_{K''(x)}) \otimes C_E(x).$$

Since $\text{Gal}^\partial(\mathcal{M}_{C_E(x)}, \eta_{C_E(x)}) = G \otimes_{C(x)} C_E(x)$, the v -curvatures of $\mathcal{M}_{K''(x)}$ must stabilise L modulo ϕ_v , in the sense of Theorem 2.14. On the other hand, L is $K'(x)$ -rational and the v -curvatures of $\mathcal{M}_{K''(x)}$ come from the v -curvatures of $\mathcal{M}_{K'(x)}$ by scalar extensions, therefore L is also stabilized by the v -curvatures of $\mathcal{M}_{K'(x)}$. This proves that $\text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)}) \otimes C(x) \subset G = \text{Stab}(L)$ and ends the proof. \square

Corollary 3.12. *Let $\mathcal{M}_{K(x)}$ be a q -difference module defined over $K(x)$. Let $U \in \text{Gl}_\nu(\text{Mer}(C^*))$ be a fundamental matrix of meromorphic solutions of $\mathcal{M}_{K(x)}$. Then,*

1. *the dimension of $\text{Gal}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the transcendence degree of the field generated by the entries of U over $C_E(x)$, i.e. the algebraic group $\text{Gal}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ measures the algebraic relations between the meromorphic solutions of $\mathcal{M}_{C_E(x)}$.*
2. *the ∂ -differential dimension of $\text{Gal}^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the differential transcendence degree of the field generated by the entries of U over $\tilde{C}_E(x)$, i.e. the differential algebraic group $\text{Gal}^\partial(\mathcal{M}_{C(x)}, \eta_{C(x)})$ gives the differential algebraic relations between the meromorphic solutions of $\mathcal{M}_{K(x)}$.*
3. *there exists a finitely generated extension K'/K such that the differential transcendence degree of the differential field generated by the entries of U over $\tilde{C}_E(x)$ is equal to the differential dimension of $\text{Gal}^\partial(\mathcal{M}_{K'(x)}, \eta_{K'(x)})$.*

Proof. 1. Proposition 3.7 and Proposition 3.11 prove that the dimension of the generic Galois group $\text{Gal}(\mathcal{M}_{C(x)}, \eta_{C(x)})$ is equal to the dimension of the group $\text{Aut}^\otimes(\mathcal{M}_{C_E(x)}, \omega_E)$ over $\tilde{C}_E(x)$, that is to the transcendence degree of the fraction field of \tilde{R}_M over $\tilde{C}_E(x)$.

2. Put together Corollary 3.9 and Proposition 3.11.

3. This is Lemma 3.10. \square

Remark 3.13. It follows from [HS08, Proposition 6.18] that there exists a one-to-one correspondence between the radical (σ_q, ∂) -ideals of the (σ_q, ∂) -Picard-Vessiot ring of the module and the differential subvarieties of the differential Galois group $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\tilde{C}_E(x)}, \tilde{\omega}_E)$. The comparison results of this section, show that this correspondence induces a correspondence between the differential subvarieties of the differential generic Galois group of \mathcal{M} and the radical (σ_q, ∂) -ideals of the (σ_q, ∂) -Picard-Vessiot generated by the meromorphic solutions of the module.

4 Specialization of the parameter q

We go back to the notation introduced in §1.3 in the case where q is transcendent over \mathbb{Q} . So we consider a field K which is a finite extension of a rational function field $k(q)$ (we recall that when speaking of differential algebraic groups, we implicitly require that k is of characteristic zero). We denote by \mathcal{P}_f the set of places of K such that the associated norms extend, up to equivalence, one of the norms of $k(q)$ attached to an irreducible polynomial $v(q) \in k[q], v(q) \neq q$, by k_v the residue field of K with respect to a place v , by ϕ_v the uniformizer of the place v and by q_v the image of q in k_v .

Let $\mathcal{M} = (M, \Sigma_q)$ be a q -difference module over an algebra \mathcal{A} of the form $\mathcal{O}_K \left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots \right]$. For almost all finite place $v \in \mathcal{P}_f$, we can consider the $k_v(x)$ -module $M_{k_v(x)} = M \otimes_{\mathcal{A}} k_v(x)$ with the structure induced by Σ_q . In this way, for almost all $v \in \mathcal{P}_f$, we obtain a q_v -difference module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$. If we can specialize modulo $q - 1$, then we get a differential module, whose connection is induced by the action of the operator $\Delta_q = \frac{\Sigma_q - Id}{(q-1)}$ on M . We call the module $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ the *specialization* of \mathcal{M} at v . It is naturally equipped with a generic Galois group $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$, associated to the forgetful functor $\eta_{k_v(x)}$ (see 1.3). Then, we can ask how the generic Galois group of the specialization $\mathcal{M}_{k_v(x)}$ is related to the specialization at the place v of the equations of the generic Galois group of \mathcal{M} . For $v \in \mathcal{C}$, Theorem 2.14 proves that one may recover $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ from the knowledge of almost all of generic Galois groups of its specializations at a cyclotomic place. In general, for $v \in \mathcal{P}_f$, the specialization of the generic Galois group gives only an upper bound for the generic Galois group of the specialized equation (see Proposition 4.15).

These problems have been studied by Y. André in [And01] where he shows, among other things, that the Picard-Vessiot groups have a nice behavior w.r.t. the specialization. Some of our results (see Proposition 4.15 for instance) are nothing more than slight adaptation of the results of André to a differential and generic context. However combined with Theorem 2.14, they lead to a description via curvatures of the generic Galois group of a differential equation (see Corollary 4.19).

4.1 Specialization of the parameter q and localization of the generic Galois group

Since specializing q we obtain both differential and q -difference modules, the best framework for studying the reduction of generic Galois groups is André's theory of generalized differential rings (*cf.* [And01, 2.1.2.1]). For clarity of exposition, we first recall some definitions and basic facts from [And01] and then deduce some results on the relation between $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ and $Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)})$ and their parametrized analogue (see Proposition 4.15). Their proof are inspired by analogous results for the local Galois group which can be found in [And01].

4.1.1 Generalized differential rings

In §4.1.1 and only in §4.1.1, we adopt a slightly more general notation.

Definition 4.1 (*cf.* [And01, 2.1.2.1]). Let R be a commutative ring with unit. A generalized differential ring (A, d) over R is an associative R -algebra A endowed with an R -derivation d from A into a left $A \otimes_R A^{op}$ -module Ω^1 . The kernel of d , denoted $Const(A)$, is called the set of constants of A .

Example 4.2.

1. Let k be a field and $k(x)$ be the field of rational functions over k . The ring $(k(x), \delta)$, with

$$\begin{aligned} \delta : k(x) &\longrightarrow \Omega^1 := dx.k(x) \\ f &\longmapsto dx.x \frac{df}{dx}, \end{aligned}$$

is a generalized differential ring over k , associated to the usual derivation $\partial := x \frac{d}{dx}$.

2. Let \mathcal{A} be a q -difference ring of the form $\mathcal{O}_K \left[x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots \right]$. The ring (\mathcal{A}, δ_q) , with

$$\begin{aligned}\delta_q : \mathcal{A} &\longrightarrow \Omega^1 := dx.\mathcal{A} \\ f &\longmapsto dx.x \frac{\sigma_q(f) - f}{(q-1)x},\end{aligned}$$

is also a generalized differential rings over \mathcal{O}_K , associated to the q -difference algebra (\mathcal{A}, σ_q) .

3. Let C denote the ring of constants of a generalized differential ring (A, d) and let I be a nontrivial proper prime ideal of C . Then the ring $A_I := A \otimes C/I$ is endowed with a structure of generalized differential ring (cf. [And01, 3.2.3.7]). In the notation of the example above, for almost any place $v \in \mathcal{P}_f$ of K , we obtain in this way a generalized differential ring of the form $(\mathcal{A} \otimes_{\mathcal{O}_K} k_v, \delta_{q_v})$.

Definition 4.3 (cf. [And01, 2.1.2.3]). A morphism of generalized differential rings $(A, d : A \mapsto \Omega^1) \mapsto (\tilde{A}, \tilde{d} : \tilde{A} \mapsto \tilde{\Omega}^1)$ is a pair $(u = u^0, u^1)$ where $u^0 : A \mapsto \tilde{A}$ is a morphism of R -algebras and u^1 is a map from Ω^1 into $\tilde{\Omega}^1$ satisfying

$$\begin{cases} u^1 \circ d = \tilde{d} \circ u^0, \\ u^1(a\omega b) = u^0(a)u^1(\omega)u^0(b), \text{ for any } a, b \in A \text{ and any } \omega \in \Omega^1. \end{cases}$$

Example 4.4. In the notation of the Example 4.2, the canonical projection $p : A \mapsto A_I$ induces a morphism u of generalized differential rings from (A, d) into (A_I, d) .

Let B be a generalized differential ring. We denote by $Diff_B$ the category of B -modules with connection (cf. [And01, 2.2]), i.e. left projective B -modules \mathcal{M} of finite type equipped with a R -linear operator

$$\nabla : \mathcal{M} \longrightarrow \Omega^1 \otimes_A \mathcal{M},$$

such that $\nabla(am) = a\nabla(m) + d(a) \otimes m$. The category $Diff_B$ is abelian, $Const(B)$ -linear, monoidal symmetric, cf. [And01, Theorem 2.4.2.2].

Example 4.5. We consider once again the different cases as in Example 4.2:

1. If $B = (k(x), \delta)$ then $Diff_B$ is the category of differential modules over $k(x)$.
2. If $B = (\mathcal{A}, \delta_q)$ then $Diff_B$ is the category of q -difference modules over \mathcal{A} . In fact, in the notation of the previous section, it is enough to set $\nabla(m) = dx.\Delta_q(m)$, for any $m \in \mathcal{M}$.

Let B be a generalized differential ring. We denote by η_B the forgetful functor from $Diff_B$ into the category of projective B -modules of finite type. For any object \mathcal{M} of $Diff_B$, we consider the forgetful functor η_B induced over the full subcategory $\langle \mathcal{M} \rangle^\otimes$ of $Diff_B$ generated by \mathcal{M} and the affine B -group-scheme $Gal(\mathcal{M}, \eta_B)$ defined over B representing the functor $Aut^\otimes(\eta_B|_{\langle \mathcal{M} \rangle^\otimes})$.

Definition 4.6. The B -scheme $Gal(\mathcal{M}, \eta_B)$ is called the *generic Galois group* of \mathcal{M} .

Let $Constr_B(\mathcal{M})$ be the collection of all constructions of linear algebra of \mathcal{M} , i.e. of all the objects of $Diff_B$ deduced from \mathcal{M} by the following B -linear algebraic constructions: direct sums, tensor products, duals, symmetric and antisymmetric products. Then one can show that $Gal(\mathcal{M}, \eta_B)$ is nothing else than the generic Galois group considered in section 2.4 (cf. [And01, 3.2.2.2]):

Proposition 4.7. *Let B be a generalized differential ring and let \mathcal{M} be an object of $Diff_B$. The affine groups scheme $Gal(\mathcal{M}, \eta_B)$ is the stabilizer inside $GL(\mathcal{M})$ of all submodules with connection of some algebraic constructions of \mathcal{M} .*

This is not the only Galois group one can define. If we assume the existence of a fiber functor ω from $Diff_B$ into the category of $Const(B)$ -module of finite type, we can define the Galois group $Aut^\otimes(\omega|_{\langle \mathcal{M} \rangle^\otimes})$ of an object \mathcal{M} as the group of tensor automorphism of the fiber functor ω restricted to $\langle \mathcal{M} \rangle^\otimes$ (cf. [And01, 3.2.1.1]). This group characterizes completely the object \mathcal{M} . For further reference, we recall the following property (cf. [And01, Theorem 3.2.2.6]):

Proposition 4.8. *The object \mathcal{M} is trivial if and only if $Aut^\otimes(\omega|_{\langle \mathcal{M} \rangle^\otimes})$ is a trivial group.*

In certain cases, the category $Diff_B$ may be endowed with a differential structure. Since $Diff_B$ is not necessarily defined over a field, we say that a category \mathcal{C} is a *differential tensor category*, if it satisfies all the axioms of [Ovc09a, Definition 3] except the assumption $End(\mathbf{1})$ is a field. We detail below the construction of the prolongation functor associated to $Diff_B$ in some precise cases.

Semi-classic situation. Let us assume that (B, ∂) is a differential subring of the differential field $(L(x), \partial := x \frac{d}{dx})$. Then $Diff_B$ is the category of differential B -modules, equivalently, of left $B[\partial]$ -modules M , free and finitely generated over B . We now define a prolongation functor F_∂ for this category as follows. If $\mathcal{M} = (M, \nabla)$ is an object of $Diff_B$ then $F_\partial(\mathcal{M}) = (M^{(1)}, \nabla)$ is the differential module defined by $M^{(1)} = B[\partial]_{\leq 1} \otimes M$, where the tensor product rule is the same one as in §2.1 (*i.e.* takes into account the Leibniz rule).

Remark 4.9. This formal definition may be expressed in a very simple and concrete way by using the differential equation attached to the module. If M is an object of $Diff_B$ given by a differential equation $\partial(Y) = AY$, the object $M^{(1)}$ is attached to the differential equation: $\partial(Z) = \begin{pmatrix} A & \partial A \\ 0 & A \end{pmatrix} Z$.

Mixed situation. Let us assume that B is a generalized differential subring of some q (resp. q_v)-difference differential field $(L(x), \delta_q)$ (resp. $(L(x), \delta_{q_v})$). The category $Diff_B$ is the category of q (resp. q_v)-difference modules. Applying the same constructions than those of Proposition 2.3, we have that $Diff_B$ is a differential tannakian category and we will denote by F_∂ its prolongation functor.

In both cases, semi-classic and mixed, we may define, as in the beginning of §2.4, the parametrized generic Galois group $Gal^\partial(\mathcal{M}, \eta_B)$ of an object \mathcal{M} of $Diff_B$. If $Constr_B^\partial$ denotes the smallest family of objects deduced from \mathcal{M} by the constructions of linear algebras and the prolongation functor F_∂ , then the parametrized analogue of Proposition 4.7 says that the differential groups scheme $Gal^\partial(\mathcal{M}, \eta_B)$ is the stabilizer inside $GL(\mathcal{M})$ of all submodules with connection of some differential constructions of \mathcal{M} .

Remark 4.10. In the semi-classic situation, the parametrized generic Galois group of a differential module \mathcal{M} is nothing else than the generic Galois group of \mathcal{M} . To see this it is enough to notice that there exists a canonical isomorphism:

$$Gal(F_\partial(\mathcal{M}), \eta_{K(x)}) \longrightarrow Gal(\mathcal{M}, \eta_{K(x)}).$$

In fact, such an arrow exists since \mathcal{M} is canonically isomorphic to a differential submodule of $F_\partial(\mathcal{M})$. Since an element $B \in Gal(\mathcal{M}, \eta_{K(x)})$ acts on $F_\partial(\mathcal{M})$ via $\begin{pmatrix} B & \partial B \\ 0 & B \end{pmatrix}$, the arrow is injective. Since an element of $Gal(\mathcal{M}, \eta_{K(x)})$ needs to be sufficiently compatible with the differential structure, it also stabilizes the differential submodules of a construction of $F_\partial(\mathcal{M})$. This last argument proves the surjectivity.

The definition below characterizes the morphisms of generalized differential rings compatible with the differential structure. We will need this notion in Lemma 4.14:

Definition 4.11 (*cf.* [And01, 2.2.2]). Let $u = (u^0, u^1) : (A, d) \mapsto (A', d')$ be a morphism of generalized differential rings. This morphism induces a tensor-compatible functor denoted by u^* from the category $Diff_A$ into the category $Diff_{A'}$. Moreover, let us assume that $Diff_A$ (resp. $Diff_{A'}$) is a differential category and let us denote by F_∂ its prolongation functor. We say that u^* is *differentially compatible* if it commutes with the prolongation functors, *i.e.* $F_\partial \circ u^* = u^* \circ F_\partial$.

4.1.2 Localization and specialization of generic Galois groups

We go back to the notation of the beginning of §4.1. We moreover assume that \mathcal{A} (resp. $\mathcal{A}_v := \mathcal{A} \otimes_{\mathcal{O}_K} k_v$) is stable under the action of ∂ . As already noticed, the q -difference algebras \mathcal{A} and \mathcal{A}_v are simple generalized differential rings (*cf.* [And01, 2.1.3.4, 2.1.3.6]). Moreover, the fraction field of \mathcal{A} (resp. \mathcal{A}_v) is $K(x)$ (resp. $k_v(x)$). If $q_v \neq 1$, the ring $(\mathcal{A}_v, \sigma_{q_v}, \partial := x \frac{d}{dx})$ (resp. the field $(k_v(x), \sigma_{q_v}, \partial := x \frac{d}{dx})$) is a q_v -difference differential ring (resp. field).

The following lemma of *localization* relates the generic (parametrized) Galois group of a module over the ring \mathcal{A} (resp. over \mathcal{A}_v) to the generic (parametrized) Galois group of its localization over the fraction field $K(x)$ (resp. $k_v(x)$) of \mathcal{A} (resp. \mathcal{A}_v). This lemma is a version of [And01, Lemma 3.2.3.6] for (parametrized) generic Galois groups.

Proposition 4.12. *Let $\mathcal{M}, \mathcal{A}, v, \mathcal{A}_v$ as above. We have*

1. $Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes K(x) \simeq Gal(\mathcal{M}_{K(x)}, \eta_{K(x)});$
2. $Gal^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes K(x) \simeq Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$
3. $Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \otimes k_v(x) \simeq Gal(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)}).$
4. $Gal^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \otimes k_v(x) \simeq Gal^{\partial}(\mathcal{M}_{k_v(x)}, \eta_{k_v(x)}).$

Remark 4.13. In the previous section we have given a description of the generic Galois group $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ via the reduction modulo ϕ_v of the operators $\Sigma_q^{\kappa_v}$. We are unable to give a similar description of $Gal(\mathcal{M}, \eta_{\mathcal{A}})$, essentially because Chevalley theorem holds only for algebraic groups over a field.

Proof. Because \mathcal{A} (resp. \mathcal{A}_v) is a simple differential ring and its fraction field $K(x)$ (resp $k_v(x)$) is semi-simple, we may apply [And01, lemma 3.2.3.6] and [And01, proposition 2.5.1.1]. We obtain that the functor

$$\begin{aligned} Loc : \langle \mathcal{M} \rangle^{\otimes, \partial} &\longrightarrow \langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial} \\ \mathcal{N} &\longmapsto \mathcal{N}_{K(x)} \end{aligned}$$

is an equivalence of monoidal categories. Moreover, Loc commutes with the prolongation functors, i.e. $F_{\partial} \circ Loc = Loc \circ F_{\partial}$. To conclude it is enough to remark that Loc also commutes with the forgetful functors. \square

So everything works quite well for the localization. Before proving some results concerning the specialization, we state an analogue of [And01, lemma 3.2.3.5] on the compatibility of constructions.

Lemma 4.14. *Let $u : (A, d) \mapsto (B, \tilde{d})$ be a morphism of integral generalized differential rings, such that B is faithfully flat over A . Then for any object \mathcal{M} of $Diff_A$ we have*

$$Constr_A(\mathcal{M}) \otimes_A B = Constr_B(\mathcal{M} \otimes_A B),$$

i.e. the constructions of linear algebra commute with the base change. If we assume moreover that $Diff_A$ and $Diff_B$ are differential tensor categories and that u^* is differentially compatible, we have

$$Constr_A^{\partial}(\mathcal{M}) \otimes_A B = Constr_B^{\partial}(\mathcal{M} \otimes_A B),$$

where $Constr^{\partial}$ denotes the construction of differential linear algebra (cf. the mixed situation in §4.1.1).

Proof. Because M is a projective A -module of finite type and B is faithfully flat over A , the canonical map $Hom_A(M, A) \otimes B \mapsto Hom_B(M \otimes B, B)$ is bijective. The first statement follows from this remark. The last one follows immediately from the first and from the definition of a differentially compatible functor (cf. Definition 4.11). \square

Finally, we have:

Proposition 4.15. *Let (\mathcal{A}, δ_q) be the generalized differential ring as in Example 4.2, (2). Let v be a finite place of K . For any \mathcal{M} object of $Diff_{\mathcal{A}}$, we have*

$$Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \subset Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{A}_v$$

and

$$Gal^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \subset Gal^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{A}_v.$$

Proof. By definition, $Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) = Aut^{\otimes}(\eta_{\mathcal{A}_v}|_{(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v)^{\otimes}})$ is the stabilizer inside $GL(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v) = GL(\mathcal{M}) \otimes_{\mathcal{A}} \mathcal{A}_v$ of the subobjects W of a construction of linear algebra of $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v$. The group $Gal(\mathcal{M}, \eta_{\mathcal{A}})$ admits a similar description. The projection map $p : \mathcal{A} \mapsto \mathcal{A}_v$ is a morphism of generalized differential rings. Since \mathcal{A}_v is faithfully flat over \mathcal{A} , we may thus apply the first part of Lemma 4.14 and we conclude that $Constr_{\mathcal{A}}(\mathcal{M}) \otimes_{\mathcal{A}} \mathcal{A}_v = Constr_{\mathcal{A}_v}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v)$ and therefore that $Gal(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}_v, \eta_{\mathcal{A}_v}) \subset Gal(\mathcal{M}, \eta_{\mathcal{A}}) \otimes_{\mathcal{A}} \mathcal{A}_v$. We give now a sketch of proof for the differential part.

If we assume that \mathcal{A} is stable under the action of ∂ then the category $\text{Diff}_{\mathcal{A}}$ is a differential tensor category as it is described in the *mixed situation* of §4.1.1 and we denote by F_{∂} its prolongation functor. Moreover, $\text{Diff}_{\mathcal{A}_v}$ is also a differential tensor category, either $q_v = 1$ and we are in the *classical situation*, either $q_v \neq 1$ and we are in the *mixed situation*. In both cases, a simple calculation shows that the projection map $p : \mathcal{A} \mapsto \mathcal{A}_v$ induces a differentially compatible functor p^* from $\text{Diff}_{\mathcal{A}}$ into $\text{Diff}_{\mathcal{A}_v}$. Then Lemma 4.14, the arguments above and the definition of the parametrized generic Galois group in terms of stabilizer of objects inside the construction of differential algebra give the last inclusion. \square

Remark 4.16. Similar results hold for differential equations (cf. [Kat90, §2.4] and [And01, §3.3]). In general one cannot obtain any semicontinuity result. In fact, the differential equation $\frac{y'}{y} = \frac{\lambda}{y}$, with λ complex parameter, has differential Galois group equal to \mathbb{C}^* . When one specializes the parameter λ on a rational value λ_0 , one gets an equation whose differential Galois group is a cyclic group of order the denominator of λ_0 . For all other values of the parameter, the Galois group is \mathbb{C}^* .

The situation appears to be more rigid for q -difference equations when q is a parameter. In fact, we can consider the q -difference equation $y(qx) = P(q)y(x)$, with $P(q) \in k(q)$. If we specialize q to a root of unity and we find a finite generic Galois group too often, we can conclude using Theorem 2.14 that $P(q) \in q^{\mathbb{Z}/r}$, for some positive integer r , and therefore that the generic Galois group of $y(qx) = P(q)y(x)$ over $K(x)$ is finite.

4.2 Upper bounds for the generic Galois group of a differential equation

Let us consider a q -difference module $\mathcal{M} = (M, \Sigma_q)$ over \mathcal{A} that admits a reduction modulo the $(q-1)$ -adic place of K , *i.e.* such that we can specialize the parameter q to 1. To simplify notation, let us denote by k_1 the residue field of K modulo $q-1$.

In this case the specialized module $\mathcal{M}_{k_1(x)} = (M_{k_1(x)}, \Delta_1)$ is a differential module. We can deduce from the results above that:

Corollary 4.17.

$$\text{Gal}(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}) \subset \text{Gal}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes k_1(x).$$

and

$$\text{Gal}^{\partial}(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}) \subset \text{Gal}^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes k_1(x).$$

Proof. Proposition 4.15 says that:

$$\text{Gal}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \subset \text{Gal}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{A}/(q-1),$$

and

$$\text{Gal}^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \subset \text{Gal}^{\partial}(\mathcal{M}, \eta_{\mathcal{A}}) \otimes \mathcal{A}/(q-1),$$

We conclude applying Proposition 4.12:

$$\text{Gal}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \otimes_{\mathcal{A}/(q-1)} k_1(x) \cong \text{Gal}(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}),$$

and

$$\text{Gal}^{\partial}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(q-1), \eta_{\mathcal{A}/(q-1)}) \otimes_{\mathcal{A}/(q-1)} k_1(x) \cong \text{Gal}^{\partial}(\mathcal{M}_{k_1(x)}, \eta_{k_1(x)}),$$

remembering that $k_1(x)$ is flat over $\mathcal{A}/(q-1)$. \square

Remark 4.18. An example of application of the theorem above is given by the “Schwartz list” for q -difference equations (cf. [DV02, Appendix]), where it is proved that the trivial basic q -difference equations are exactly the deformation of the trivial Gauss hypergeometric differential equations.

The Schwartz list for higher order basic hypergeometric equations has been established by J. Roques (cf. [Roq09, §8]), and is another example of this phenomenon.

On the other hand, given a $k(x)/k$ -differential module (M, ∇) , we can fix a basis \underline{e} of M such that

$$\nabla(\underline{e}) = \underline{e}G(x),$$

where we have identified ∇ with $\nabla(\frac{d}{dx})$. The horizontal vectors for ∇ are solutions of the system $Y'(x) = -G(x)Y(x)$. Then, if $K/k(q)$ is a finite extension, we can define a natural q -difference module structure over $M_{K(x)} = M \otimes_{k(x)} K(x)$ setting

$$\Sigma_q(\underline{e}) = \underline{e}(1 + (q-1)xG(x)),$$

and extending the action of Σ_q to $M_{K(x)}$ by semi-linearity. The definition of Σ_q depends on the choice of \underline{e} , so that we should rather write $\Sigma_q^{(\underline{e})}$, which we avoid to not complicate the notation. Thus, starting from a differential module M we may find a q -difference module $M_{K(x)}$ such that M is the specialization of $M_{K(x)}$ at the place of K defined by $q = 1$. The q -deformation we have considered here is somehow a little bit trivial and does not correspond for instance to the process used to deform a hypergeometric differential equation into a q -hypergeometric equation. Anyway, we just want to show that a q -deformation combined with our results gives an arithmetic description of the generic Galois group of a differential equation. This description depends obviously of the process of q -deformation and its refinement is strongly related to the sharpness of the q -deformation used.

Using the “trivial” q -deformation, we have the following description

Corollary 4.19. *The generic Galois group of (M, ∇) is contained in the “specialization at $q = 1$ ” of the smallest algebraic subgroup of $Gl(M_{K(x)})$ containing the reduction modulo ϕ_v of $\Sigma_q^{\kappa_v}$:*

$$\Sigma_q^{\kappa_v} \underline{e} = \underline{e} \prod_{i=0}^{\kappa_v-1} (1 + (q-1)q^i x G(q^i x))$$

for almost all $v \in \mathcal{C}_K$.

Corollary 4.20. *Suppose that k is algebraically closed. Then a differential module (M, ∇) is trivial over $k(x)$ if and only if there exists a basis \underline{e} such that $\nabla(\underline{e}) = \underline{e}G(x)$ and for almost all primitive roots of unity ζ in a fixed algebraic closure \bar{k} of k we have:*

$$\left[\prod_{i=0}^{n-1} (1 + (q-1)q^i x G(q^i x)) \right]_{q=\zeta} = \text{identity matrix},$$

where n is the order of ζ .

Proof. If the identity above is verified, then the Galois group of (M, ∇) is trivial, which implies that (M, ∇) is trivial over $k(x)$. On the other hand, if (M, ∇) is trivial over $k(x)$, there exists a basis \underline{e} of M over $k(x)$ such that $\nabla(\underline{e}) = 0$. This ends the proof. \square

References

- [And01] Y. André. Différentielles non commutatives et théorie de Galois différentielle ou aux différences. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 34(5):685–739, 2001.
- [BM96] M. Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Mathematics*, 154(1-3):1–25, 1996.
- [BMF95] M. Bousquet-Mélou and J.-M. Fédou. The generating function of convex polyominoes: the resolution of a q -differential system. *Discrete Mathematics*, 137(1-3):53–75, 1995.
- [BMP03] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoretical Computer Science*, 307(2):257–276, 2003. Random generation of combinatorial objects and bijective combinatorics.
- [Cas72] P.J. Cassidy. Differential algebraic groups. *American Journal of Mathematics*, 94:891–954, 1972.

- [CHS08] Z. Chatzidakis, C. Hardouin, and M.F. Singer. On the definitions of difference Galois groups. In *Model Theory with applications to algebra and analysis, I and II*, pages 73–109. Cambridge University Press, 2008.
- [CS06] P.J. Cassidy and M.F. Singer. Galois theory of parameterized differential equations and linear differential algebraic groups. In *Differential Equations and Quantum Groups*, volume 9 of *IRMA Lectures in Mathematics and Theoretical Physics*, pages 113–157. 2006.
- [Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol II*, volume 87 of *Prog.Math.*, pages 111–195. Birkhäuser, Boston, 1990.
- [DV02] L. Di Vizio. Arithmetic theory of q -difference equations. The q -analogue of Grothendieck-Katz’s conjecture on p -curvatures. *Inventiones Mathematicae*, 150(3):517–578, 2002. arXiv:math.NT/0104178.
- [DVH10a] L. Di Vizio and C. Hardouin. Algebraic and differential generic Galois groups for q -difference equations, followed by the appendix "The Galois D -groupoid of a q -difference system" by Anne Granier. Arxiv:1002.4839v4, unpublished, 2010.
- [DVH10b] L. Di Vizio and C. Hardouin. Courbures, groupes de Galois génériques et D -groupoïde de Galois d’un système aux q -différences. *Comptes Rendus Mathématique*, 348(17–18):951–954, 2010.
- [DVH11a] L. Di Vizio and C. Hardouin. On the Grothendieck conjecture on p -curvatures for q -difference equations. 2011.
- [DVH11b] L. Di Vizio and C. Hardouin. Parameterized generic Galois groups for q -difference equations, followed by the appendix "The Galois D -groupoid of a q -difference system" by Anne Granier. 2011.
- [DVH11c] L. Di Vizio and C. Hardouin. Descent for differential Galois theory of difference equations. Confluence and q -dependency. To appear in *Pacific Journal of Mathematics*. Arxiv:1002.4839v4, 2011.
- [DVRSZ03] L. Di Vizio, J.-P. Ramis, J. Sauloy, and C. Zhang. Équations aux q -différences. *Gazette des Mathématiciens*, 96:20–49, 2003.
- [GGO11] H. Gillet, S. Gorchinskiy, and A. Ovchinnikov. Parameterized Picard-Vessiot extensions and Atiyah extensions, 2011.
- [GM93] M. Granger and P. Maisonobe. Differential modules. In *D -modules cohérents et holonomes sous la direction de P. Maisonobe et C. Sabbah*, pages 103–167. Hermann, 1993.
- [Gra] A. Granier. A Galois D -groupoid for q -difference equations. *Annales de l’Institut Fourier*. To appear.
- [Gra10] A. Granier. Un D -groupoïde de Galois local pour les systèmes aux q -différences fuchsien. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 348(5-6):263–265, 2010.
- [Har08] C. Hardouin. Hypertranscendance des systèmes aux différences diagonaux. *Compositio Mathematica*, 144(3):565–581, 2008.
- [HS08] C. Hardouin and M.F. Singer. Differential Galois theory of linear difference equations. *Mathematische Annalen*, 342(2):333–377, 2008.
- [Kam] M. Kamensky. Tannakian formalism over fields with operators. arXiv:1111.7285.
- [Kat82] N. M. Katz. A conjecture in the arithmetic theory of differential equations. *Bulletin de la Société Mathématique de France*, 110(2):203–239, 1982.

- [Kat90] N. M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, 1990.
- [Kol74] E. R. Kolchin. Constrained extensions of differential fields. *Advances in Mathematics*, 12:141–170, 1974.
- [LY08] B.Q. Li and Z. Ye. On differential independence of the Riemann zeta function and the Euler gamma function. *Acta Arithmetica*, 135(4):333–337, 2008.
- [Mal09] B. Malgrange. Pseudogroupes de lie et théorie de galois différentielle. preprint, 2009.
- [Mar07] L. Markus. Differential independence of Γ and ζ . *Journal of Dynamics and Differential Equations*, 19(1):133–154, 2007.
- [MO10] A. Minchenko and A. Ovchinnikov. Zariski Closures of Reductive Linear Differential Algebraic Groups. arXiv:1005.0042, 2010.
- [Ovc08] A. Ovchinnikov. Tannakian approach to linear differential algebraic groups. *Transformation Groups*, 13(2):413–446, 2008.
- [Ovc09a] A. Ovchinnikov. Differential tannakian categories. *Journal of Algebra*, 321(10):3043–3062, 2009.
- [Ovc09b] A. Ovchinnikov. Tannakian categories, linear differential algebraic groups, and parametrized linear differential equations. *Transformation Groups*, 14(1):195–223, 2009.
- [PN09] A. Peón Nieto. On sigma-delta-Picard-Vessiot extensions. To appear in *Communications in Algebra*, 2009.
- [Pra86] C. Praagman. Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems. *Journal für die Reine und Angewandte Mathematik*, 369:101–109, 1986.
- [Ram92] J.-P. Ramis. About the growth of entire functions solutions of linear algebraic q -difference equations. *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6*, 1(1):53–94, 1992.
- [Roq09] J. Roques. Generalized basic hypergeometric equations. *Preprint Université de Grenoble.*, pages 1–35, 2009.
- [Sau00] J. Sauloy. Systèmes aux q -différences singuliers réguliers: classification, matrice de connexion et monodromie. *Annales de l’Institut Fourier*, 50(4):1021–1071, 2000.
- [Sau04a] J. Sauloy. Algebraic construction of the Stokes sheaf for irregular linear q -difference equations. *Astérisque*, 296:227–251, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I.
- [Sau04b] J. Sauloy. Galois theory of Fuchsian q -difference equations. *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série*, 36(6):925–968, 2004.
- [SR72] N. Saavedra Rivano. *Catégories tannakiennes*, volume 265 of *Lecture Notes in Mathematics*. Springer, Berlin, 1972.
- [vdPR07] Marius van der Put and Marc Reversat. Galois theory of q -difference equations. *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6*, 16(3):665–718, 2007.
- [vdPS97] M. van der Put and M.F. Singer. *Galois theory of difference equations*. Springer-Verlag, Berlin, 1997.
- [Wib11] M. Wibmer. Existence of ∂ -parameterized Picard-Vessiot extensions over fields with algebraically closed constants. ArXiv:1104.3514, 2011.